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Peridynamic States

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Objective of peridynamics

Some limitations of the standard theory of solid mechanics

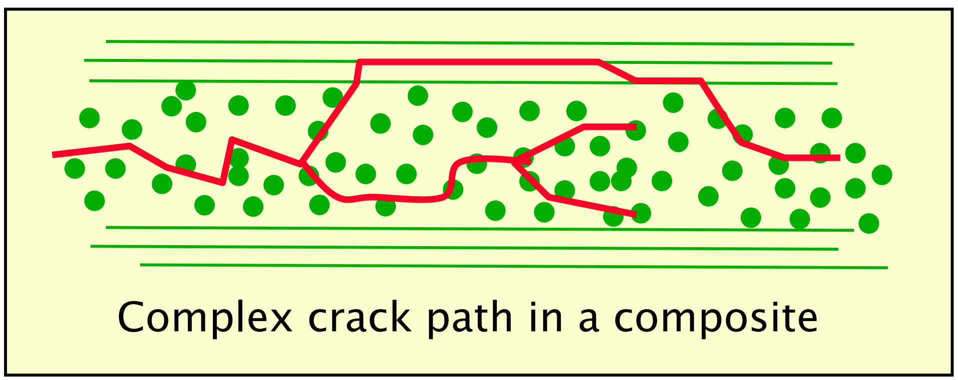
- It is incompatible with the essential physical nature of particles and cracks.
  - Can't apply the PDEs directly.
- Can't easily include long-range interactions.

What the peridynamic theory seeks to do

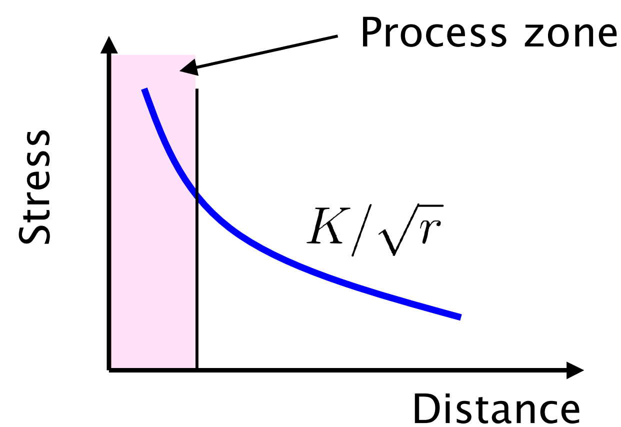
- To predict the mechanics of continuous and discontinuous media with mathematical consistency.
  - Everything should emerge from the same continuum model.

Cracks vs. continua: Why this issue is important

- Typical approaches require some fix at the discretized level.
- LEFM adds extra laws that tell a crack what to do.
  - These laws are known only in idealized cases.



Complex crack path in a composite



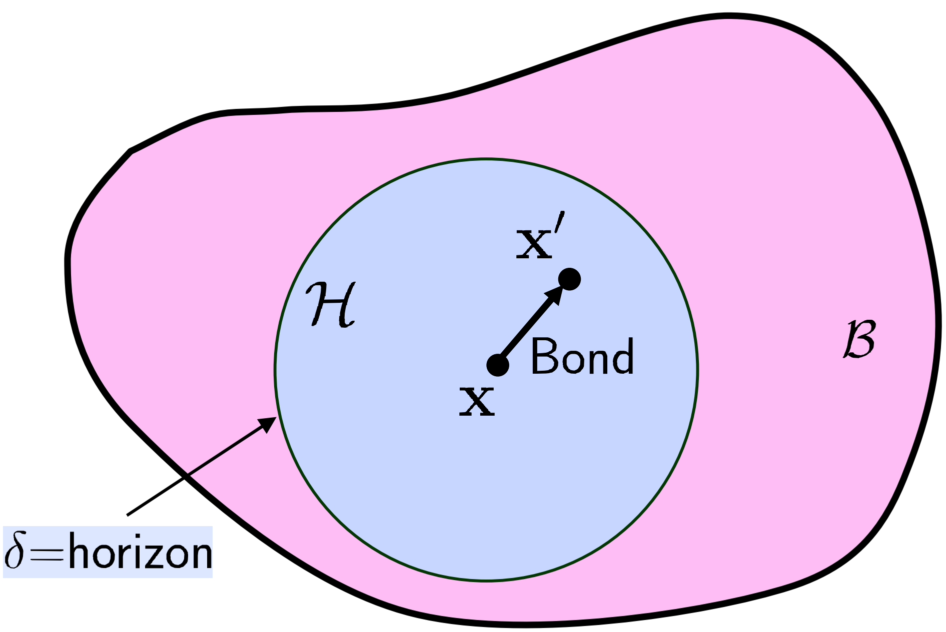
The reality of fracture may be too complex to represent in the form

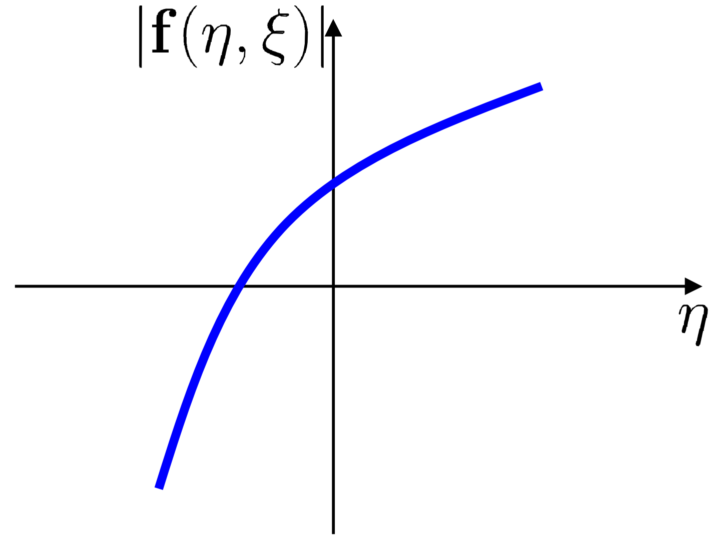
$$\dot{a} = f(K)$$

Original concept (2000): Continuum as a network of bonds

- Any point $\mathbf{x}$ interacts directly with other points within a finite distance δ called the "horizon." Equation of motion:

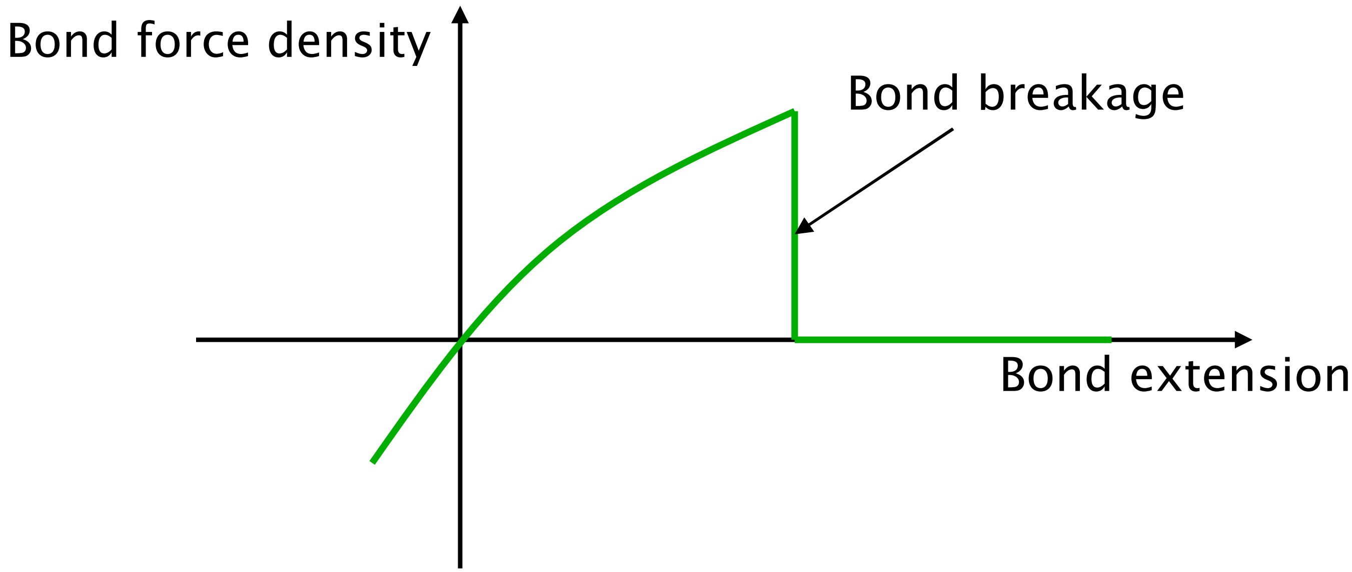
$$\rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x},t) = \int_{\mathcal{H}} \mathbf{f}(\mathbf{u}(\mathbf{x}',t) - \mathbf{u}(\mathbf{x},t), \mathbf{x}' - \mathbf{x})\, dV_{\mathbf{x}'} + \mathbf{b}(\mathbf{x},t)$$



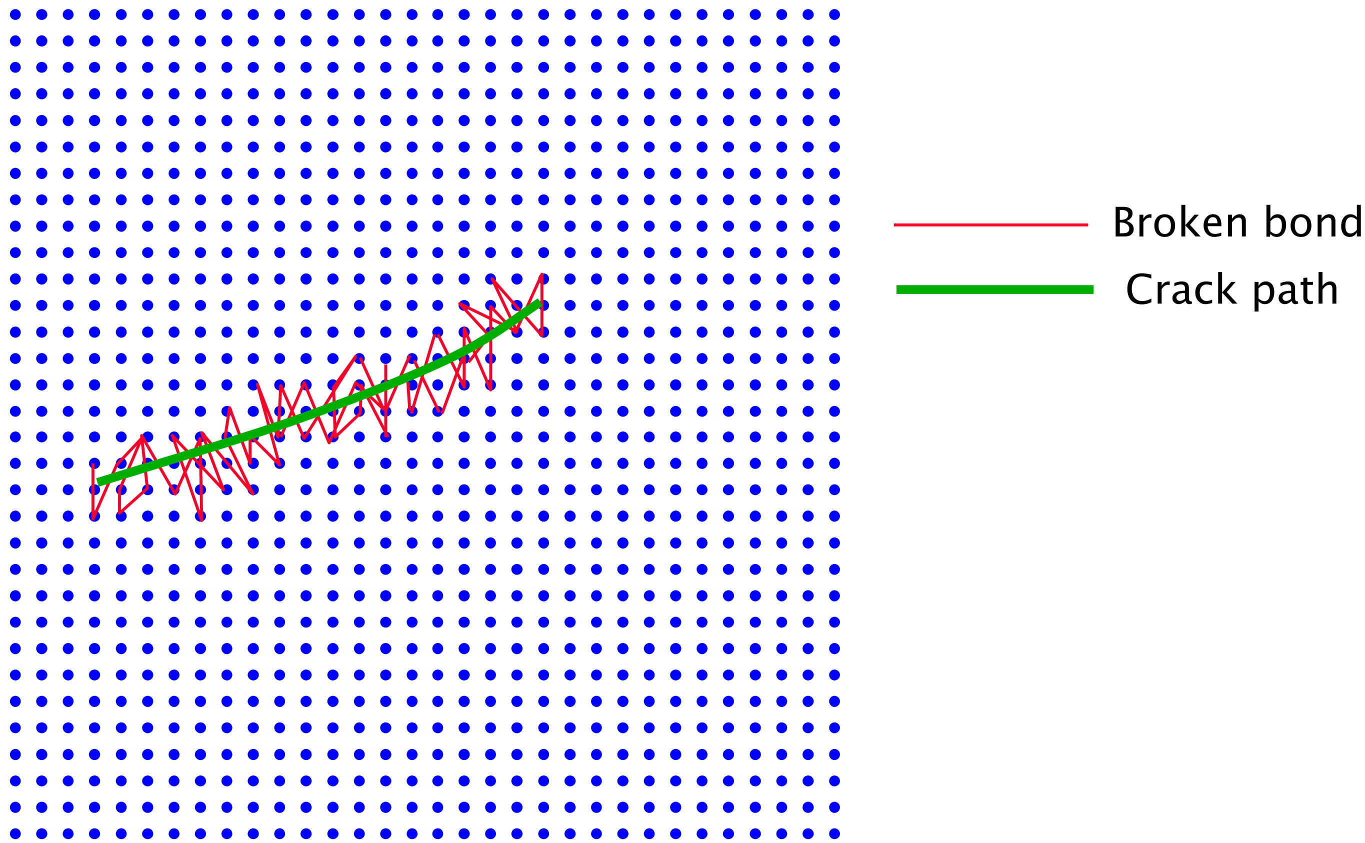


How damage and fracture are modeled

- Bonds can break irreversibly according to some criterion.
- Broken bonds carry no force.



Bond breakage forms cracks “autonomously”

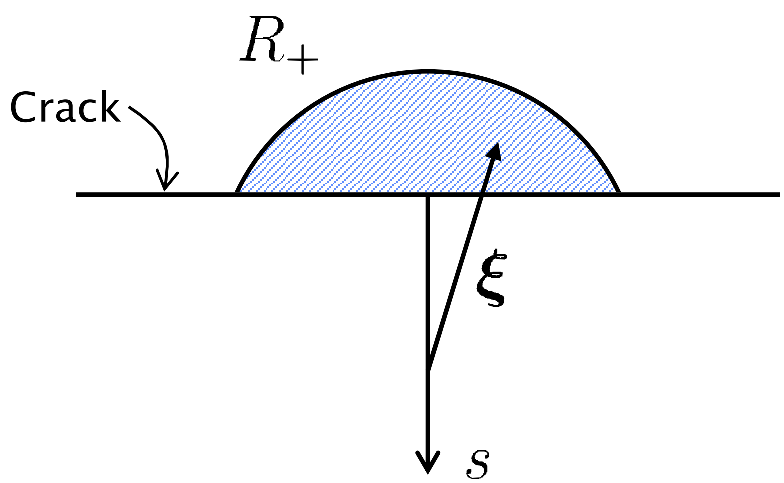


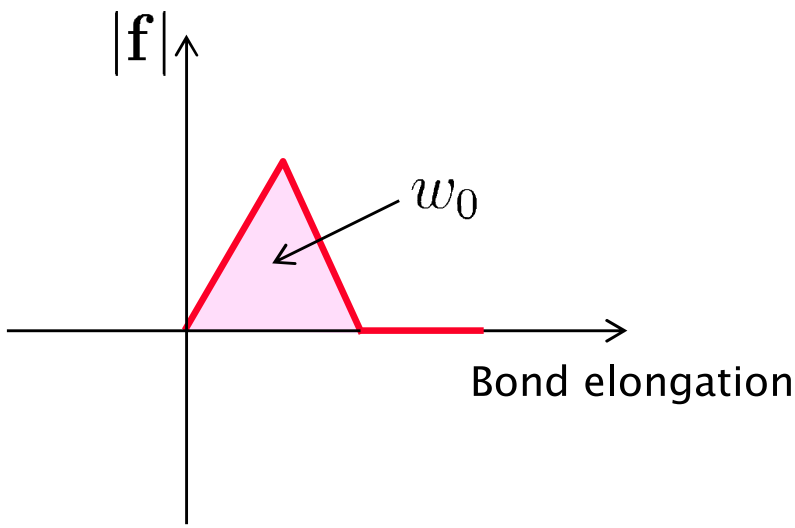
When a bond breaks, its load is shifted to its neighbors, leading to progressive failure.

Energy balance for an advancing crack

If the work required to break the bond $\boldsymbol{\xi}$ is $w_0(\boldsymbol{\xi})$, then the energy release rate is found by summing this work per unit crack area (J. Foster):

$$G = \int_0^\delta \int_{R_+} w_0(\boldsymbol{\xi})\, dV_{\boldsymbol{\xi}}\, ds$$





There is also a version of the J-integral that applies in this theory.

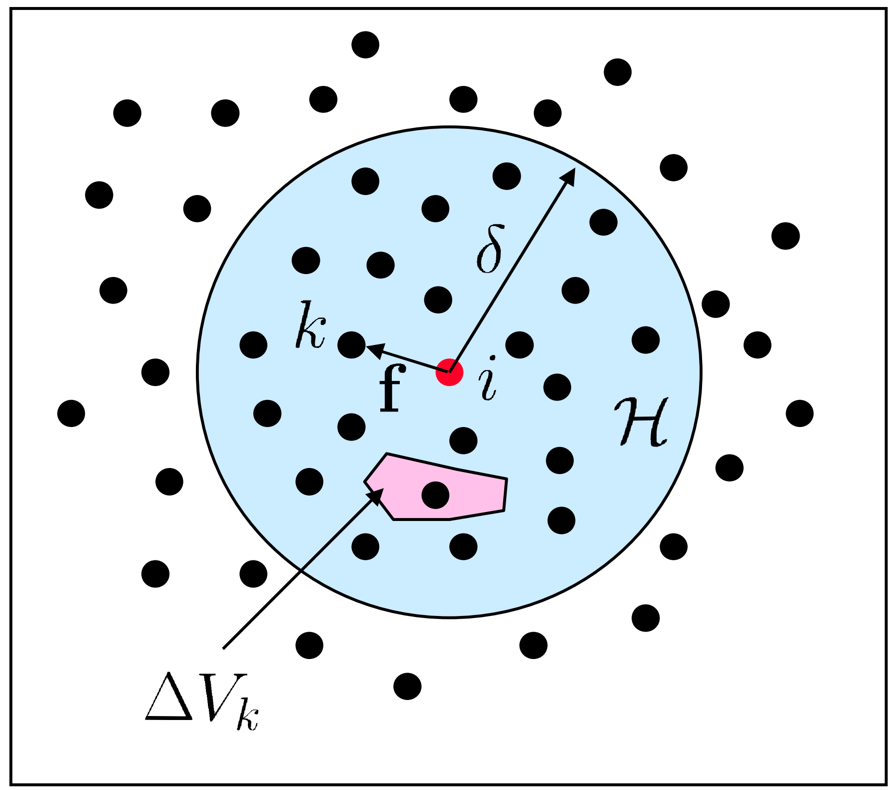
EMU numerical method

- Integral is replaced by a finite sum: resulting method is meshless and Lagrangian.

$$\rho \ddot{\mathbf{y}}(\mathbf{x}, t) = \int_{\mathcal{H}} \mathbf{f}(\mathbf{x}', \mathbf{x}, t)\, dV_{\mathbf{x}'} + \mathbf{b}(\mathbf{x}, t)$$

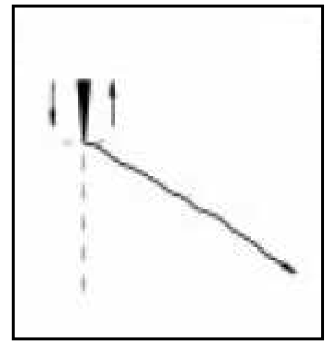
$$\downarrow$$

$$\rho \ddot{\mathbf{y}}_i^n = \sum_{k \in \mathcal{H}} \mathbf{f}(\mathbf{x}_k, \mathbf{x}_i, t)\, \Delta V_k + \mathbf{b}_i^n$$



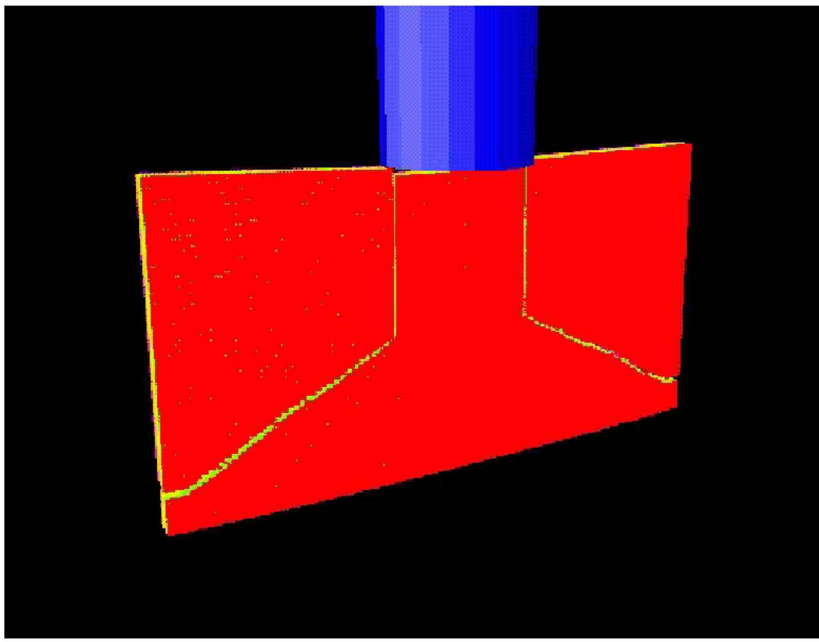
Dynamic fracture in a hard steel plate

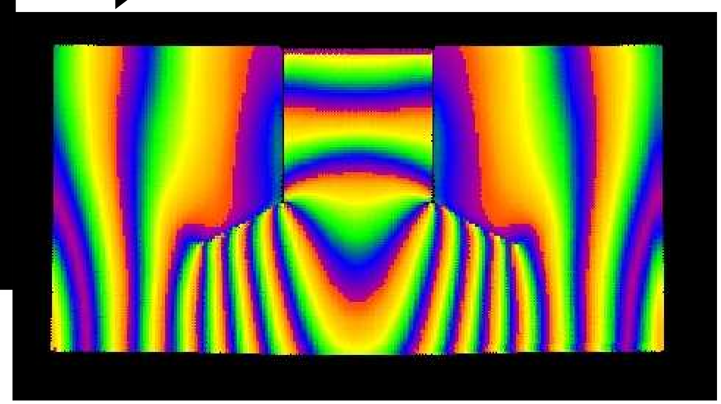
- Dynamic fracture in maraging steel (Kalthoff & Winkler, 1988)
  - Mode-II loading at notch tips results in mode-I cracks at 70deg angle.
  - 3D EMU model reproduces the crack angle.



Experiment

EMU*





S. A. Silling, Dynamic fracture modeling with a meshfree peridynamic code, in *Computational Fluid and Solid Mechanics 2003*, K.J. Bathe, ed., Elsevier, pp. 641-644.

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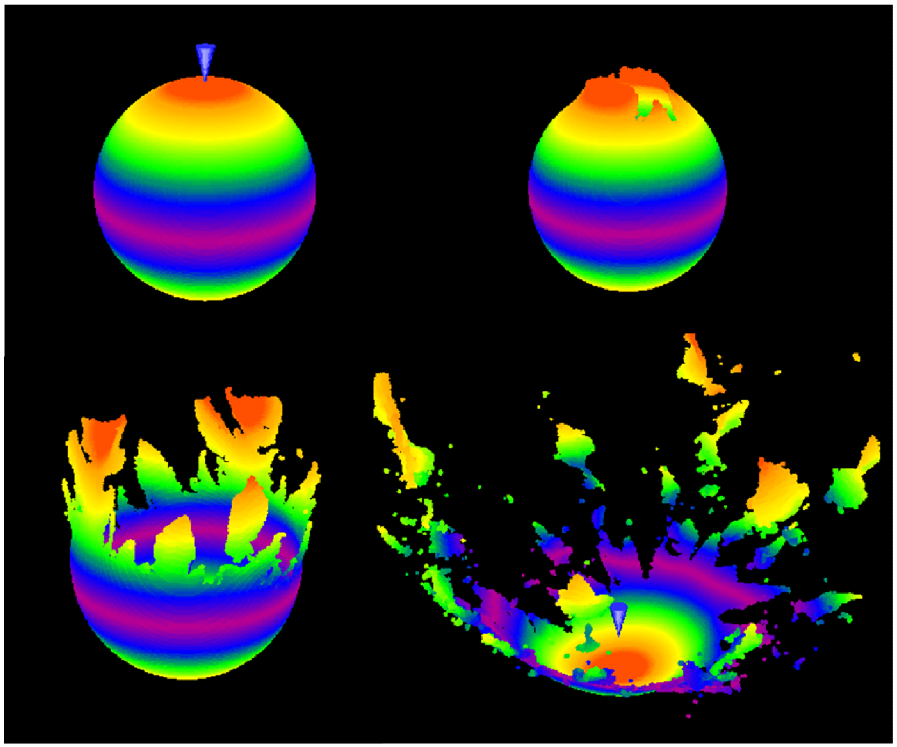
Dynamic fracture in membranes

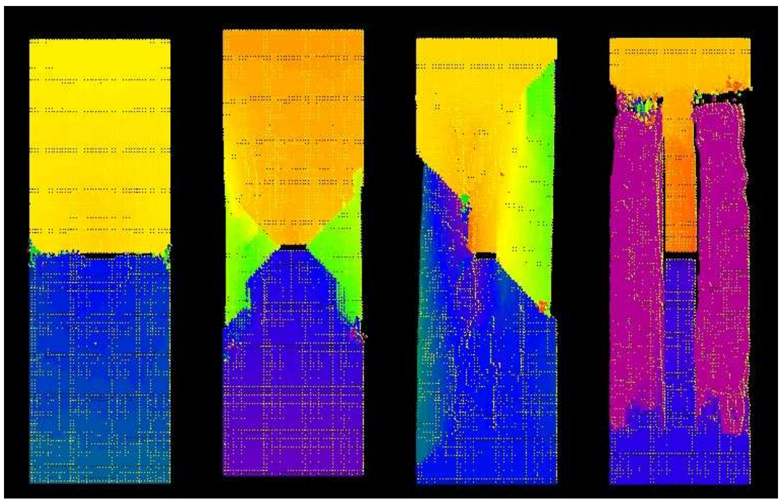


EMU model of a balloon penetrated
by a fragment

Early high speed photograph by Harold Edgerton
(MIT collection)
http://mit.edu/6.933/www/Fall2000/edgerton/edgerton.ppt

Splitting and fracture mode change in composites

- Distribution of fiber directions between plies strongly influences the way cracks grow.



EMU simulations for different layups

Typical crack growth in a notched laminate
(photo courtesy Boeing)

Limitations of the original theory

- Pair interactions imply Poisson ratio = 1/4.
- Can’t use traditional stress-strain models.
- Can’t enforce plastic incompressibility.

New approach

- Retain idea of bond forces.
- But bond forces depend on the collective deformation of the family.

Some references

• S.A. Silling, M. Epton, O. Weckner, J. Xu, and E. Askari, Peridynamic states and constitutive modeling, J. Elast. 88 (2007) 151–184
DOI: 10.1007/s10659-007-9125-1

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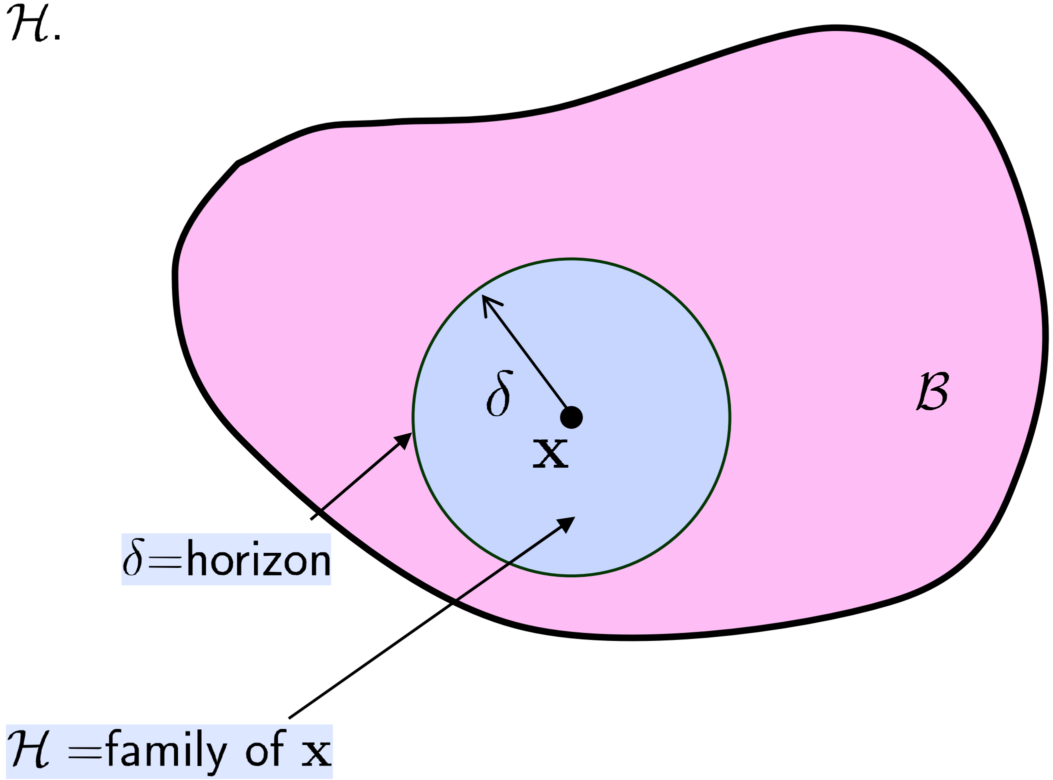
• S.A. Silling and R. B. Lehoucq, Convergence of peridynamics to classical elasticity theory, J. Elast. 93 (2008) 13–37
DOI: 10.1007/s10659-008-9163-3

• S.A. Silling, Linearized theory of peridynamic states, J. Elast. (2010, to appear)
DOI: 10.1007/s10659-009-9234-0

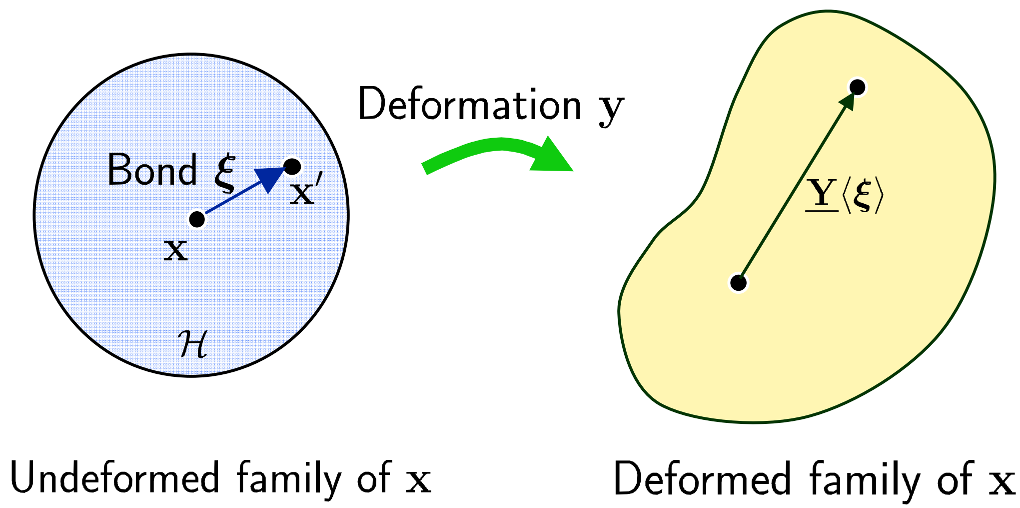
Peridynamics basics: Horizon and family

- Any point $\mathbf{x}$ interacts directly with other points within a finite distance δ called the "horizon."
- The material within a distance δ of $\mathbf{x}$ is called the "family" of $\mathbf{x}$, $\mathcal{H}$.



Why we need states

We want to express the idea that the strain energy density at $\mathbf{x}$ depends **collectively** on the deformation of the family of $\mathbf{x}$.



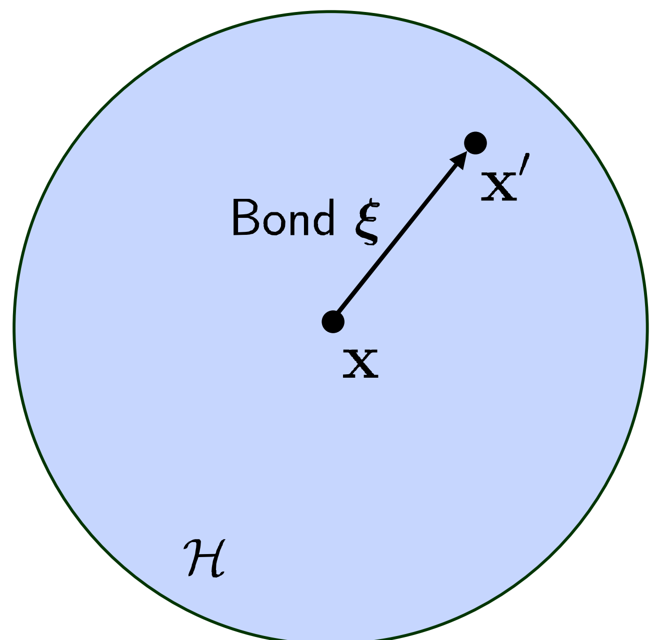
Standard:	Peridynamic:
$W\left(\dfrac{\partial \mathbf{y}}{\partial \mathbf{x}}\right)$	$W(\underline{\mathbf{Y}})$

Definition of a state

- A state is a function on $\mathcal{H}$.
- A vector state $\underline{\mathbf{A}}$ maps each bond $\boldsymbol{\xi} \in \mathcal{H}$ to a vector written $\underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle$.
- Scalar states: $\underline{A}\langle\boldsymbol{\xi}\rangle$ is scalar valued.
- Double states map pairs of bonds to second order tensors: $\underline{\mathbb{A}}\langle\boldsymbol{\xi},\boldsymbol{\zeta}\rangle$.

$$\xi = \mathbf{x}' - \mathbf{x}$$

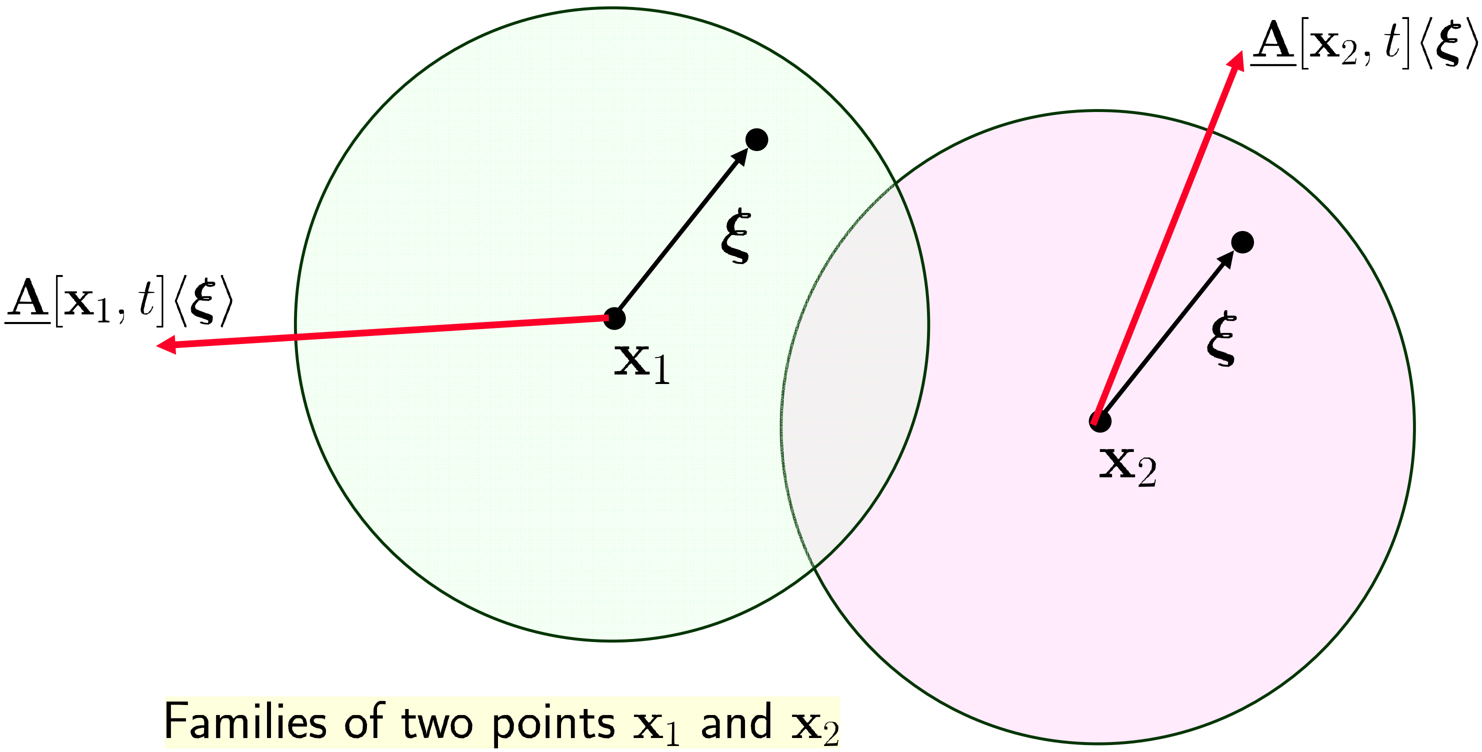
Bonds are defined in the reference configuration.



State fields

- States can depend on position (in the reference configuration) and time.

$$\underline{\mathbf{A}}[\mathbf{x}, t]\langle \boldsymbol{\xi} \rangle$$



Families of two points $\mathbf{x}_1$ and $\mathbf{x}_2$

Dot product of two states

- Suppose $\underline{\mathbf{A}}$ and $\underline{\mathbf{B}}$ are vector states.
- Define a scalar called the dot product of $\underline{\mathbf{A}}$ and $\underline{\mathbf{B}}$ by

$$\underline{\mathbf{A}} \bullet \underline{\mathbf{B}} = \int_{\mathcal{H}} \underline{\mathbf{A}}\langle \boldsymbol{\xi} \rangle \cdot \underline{\mathbf{B}}\langle \boldsymbol{\xi} \rangle \, dV_{\boldsymbol{\xi}}.$$

- In components,

$$\underline{\mathbf{A}} \bullet \underline{\mathbf{B}} = \int_{\mathcal{H}} \underline{A}_i\langle \boldsymbol{\xi} \rangle \underline{B}_i\langle \boldsymbol{\xi} \rangle \, dV_{\boldsymbol{\xi}}.$$

- Norm of a vector state:

$$||\underline{\mathbf{A}}|| = \sqrt{\underline{\mathbf{A}} \bullet \underline{\mathbf{A}}}$$

Dot product of two states, ctd.

- Suppose $\underline{a}$ and $\underline{b}$ are scalar states.

$$\underline{a} \bullet \underline{b} = \int_{\mathcal{H}} \underline{a}\langle \boldsymbol{\xi} \rangle \underline{b}\langle \boldsymbol{\xi} \rangle \; dV_{\boldsymbol{\xi}}.$$

- *Point product* is a scalar state:

$$(\underline{ab})\langle \boldsymbol{\xi} \rangle = \underline{a}\langle \boldsymbol{\xi} \rangle \underline{b}\langle \boldsymbol{\xi} \rangle$$

Functions of states, Frechet derivatives

- Let $\Psi(\underline{\mathbf{A}})$ be a scalar valued function of a vector state.
- How much does Ψ change if we change $\underline{\mathbf{A}}$? Suppose there is a vector state $\Psi_{\underline{\mathbf{A}}}$ such that

$$\Psi(\underline{\mathbf{A}} + \underline{\mathbf{a}}) = \Psi(\underline{\mathbf{A}}) + \Psi_{\underline{\mathbf{A}}} \bullet \underline{\mathbf{a}} + o(||\underline{\mathbf{a}}||)$$

  for any vector state $\underline{\mathbf{a}}$.

Less than first order.

- $\Psi_{\underline{\mathbf{A}}}(\underline{\mathbf{A}})$ is the Fréchet derivative of Ψ at $\underline{\mathbf{A}}$.

Maurice Rene Frechet

Concept is similar to the gradient in $\mathbb{R}^3$, e.g.,

$$f(\mathbf{x} + \delta\mathbf{x}) = f(\mathbf{x}) + f_{\mathbf{x}}(\mathbf{x}) \cdot \delta\mathbf{x} + o(|\delta\mathbf{x}|)$$

except that $\underline{\mathbf{A}}$ lives in an infinite dimensional space.

Frechet derivative examples: 1

- Find the Frechet derivative of $\Psi(\underline{\mathbf{A}}) = \underline{\mathbf{A}} \bullet \underline{\mathbf{A}}$:

$$\begin{aligned}
\Psi(\underline{\mathbf{A}} + \underline{\mathbf{a}}) &= (\underline{\mathbf{A}} + \underline{\mathbf{a}}) \bullet (\underline{\mathbf{A}} + \underline{\mathbf{a}}) \\
&= \int (\underline{\mathbf{A}}\langle \boldsymbol{\xi} \rangle + \underline{\mathbf{a}}\langle \boldsymbol{\xi} \rangle) \cdot (\underline{\mathbf{A}}\langle \boldsymbol{\xi} \rangle + \underline{\mathbf{a}}\langle \boldsymbol{\xi} \rangle)\, dV_{\boldsymbol{\xi}} \\
&= \underline{\mathbf{A}} \bullet \underline{\mathbf{A}} + 2 \int \underline{\mathbf{A}}\langle \boldsymbol{\xi} \rangle \cdot \underline{\mathbf{a}}\langle \boldsymbol{\xi} \rangle\, dV_{\boldsymbol{\xi}} + O(||\underline{\mathbf{a}}||) \\
&= \Psi(\underline{\mathbf{A}}) + 2\underline{\mathbf{A}} \bullet \underline{\mathbf{a}} + O(||\underline{\mathbf{a}}||)
\end{aligned}$$

Therefore

$$\Psi_{\underline{\mathbf{A}}} = 2\underline{\mathbf{A}}.$$

Frechet derivative examples: 2

- Find the Frechet derivative of $\Psi(\underline{\mathbf{A}}) = \int |\underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle| \, dV_{\boldsymbol{\xi}}$:

$$
\begin{aligned}
\Psi(\underline{\mathbf{A}} + \underline{\mathbf{a}}) &= \int |\underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle + \underline{\mathbf{a}}\langle\boldsymbol{\xi}\rangle| \, dV_{\boldsymbol{\xi}} \\
&= \int \sqrt{(\underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle + \underline{\mathbf{a}}\langle\boldsymbol{\xi}\rangle)\cdot(\underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle + \underline{\mathbf{a}}\langle\boldsymbol{\xi}\rangle)} \, dV_{\boldsymbol{\xi}} \\
&= \int |\underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle| \sqrt{1 + \frac{2\underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle \cdot \underline{\mathbf{a}}\langle\boldsymbol{\xi}\rangle}{|\underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle|^2} + \dots} \, dV_{\boldsymbol{\xi}} \\
&= \int |\underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle| \left(1 + \frac{\underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle \cdot \underline{\mathbf{a}}\langle\boldsymbol{\xi}\rangle}{|\underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle|^2} + \dots\right) dV_{\boldsymbol{\xi}} \\
&= \Psi(\underline{\mathbf{A}}) + \int \frac{\underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle \cdot \underline{\mathbf{a}}\langle\boldsymbol{\xi}\rangle}{|\underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle|} \, dV_{\boldsymbol{\xi}} + \dots
\end{aligned}
$$

Therefore

$$
\Psi_{\underline{\mathbf{A}}} = \frac{\underline{\mathbf{A}}}{|\underline{\mathbf{A}}|}.
$$

Frechet derivative examples: 3

- Find the Frechet derivative of $\Psi(\underline{\mathbf{A}}) = \int \underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle \cdot \underline{\mathbf{A}}\langle\beta\boldsymbol{\xi}\rangle \; dV_{\boldsymbol{\xi}}$ where β is a constant:

$$\Psi(\underline{\mathbf{A}}+\underline{\mathbf{a}}) = \int (\underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle + \underline{\mathbf{a}}\langle\boldsymbol{\xi}\rangle) \cdot (\underline{\mathbf{A}}\langle\beta\boldsymbol{\xi}\rangle + \underline{\mathbf{a}}\langle\beta\boldsymbol{\xi}\rangle) \, dV_{\boldsymbol{\xi}}$$

$$= \Psi(\underline{\mathbf{A}}) + \int (\underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle \cdot \underline{\mathbf{a}}\langle\beta\boldsymbol{\xi}\rangle + \underline{\mathbf{A}}\langle\beta\boldsymbol{\xi}\rangle \cdot \underline{\mathbf{a}}\langle\boldsymbol{\xi}\rangle) \, dV_{\boldsymbol{\xi}} + \dots$$

$\boldsymbol{\zeta} = \beta\boldsymbol{\xi}$

$$= \Psi(\underline{\mathbf{A}}) + \int \underline{\mathbf{A}}\langle\beta^{-1}\boldsymbol{\zeta}\rangle \cdot \underline{\mathbf{a}}\langle\boldsymbol{\zeta}\rangle(\beta^{-3} \, dV_{\boldsymbol{\zeta}}) + \int \underline{\mathbf{A}}\langle\beta\boldsymbol{\xi}\rangle \cdot \underline{\mathbf{a}}\langle\boldsymbol{\xi}\rangle \, dV_{\boldsymbol{\xi}} + \dots$$

$$= \Psi(\underline{\mathbf{A}}) + \int \left(\beta^{-3}\underline{\mathbf{A}}\langle\beta^{-1}\boldsymbol{\xi}\rangle + \underline{\mathbf{A}}\langle\beta\boldsymbol{\xi}\rangle\right) \cdot \underline{\mathbf{a}}\langle\boldsymbol{\xi}\rangle \, dV_{\boldsymbol{\xi}} + \dots$$

Therefore

$$\Psi_{\underline{\mathbf{A}}}\langle\boldsymbol{\xi}\rangle = \beta^{-3}\underline{\mathbf{A}}\langle\beta^{-1}\boldsymbol{\xi}\rangle + \underline{\mathbf{A}}\langle\beta\boldsymbol{\xi}\rangle.$$

Frechet derivative examples: 4

- Find the Frechet derivative of $\Psi(\underline{\mathbf{A}}) = \mathbf{c} \cdot \underline{\mathbf{A}}\langle\boldsymbol{\xi}_0\rangle$ where $\mathbf{c}$ is a constant vector and $\boldsymbol{\xi}_0 \in \mathcal{H}$ is a given bond:

$$\begin{aligned}\Psi(\underline{\mathbf{A}} + \underline{\mathbf{a}}) &= \int \mathbf{c} \cdot (\underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle + \underline{\mathbf{a}}\langle\boldsymbol{\xi}\rangle)\Delta(\boldsymbol{\xi} - \boldsymbol{\xi}_0)\, dV_{\boldsymbol{\xi}} \\ &= \Psi(\underline{\mathbf{A}}) + \int \Delta(\boldsymbol{\xi} - \boldsymbol{\xi}_0)\mathbf{c} \cdot \underline{\mathbf{a}}\langle\boldsymbol{\xi}\rangle\, dV_{\boldsymbol{\xi}}\end{aligned}$$

where Δ is the Dirac delta function. Therefore

$$\Psi_{\underline{\mathbf{A}}}\langle\boldsymbol{\xi}\rangle = \Delta(\boldsymbol{\xi} - \boldsymbol{\xi}_0)\mathbf{c}.$$

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Frechet derivative examples: 5

- Find the Frechet derivative of $\Psi(\underline{\mathbf{A}}) = \int f(\underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle)\; dV_{\boldsymbol{\xi}}$ where $f(\mathbf{v})$ is a scalar-valued function of a vector:

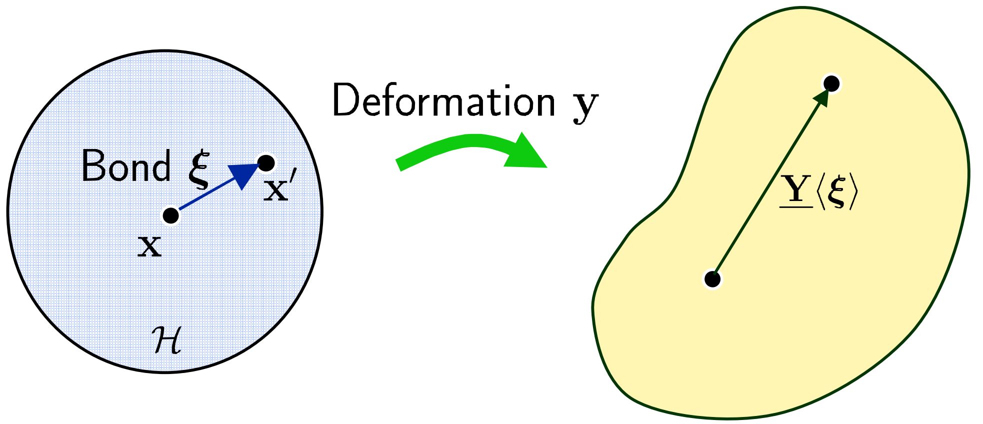
$$
\begin{aligned}
\Psi(\underline{\mathbf{A}} + \underline{\mathbf{a}}) &= \int f(\underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle + \underline{\mathbf{a}}\langle\boldsymbol{\xi}\rangle)\, dV_{\boldsymbol{\xi}} \\
&= \int \big(f(\underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle) + \operatorname{grad} f(\underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle) \cdot \underline{\mathbf{a}}\langle\boldsymbol{\xi}\rangle\big)\, dV_{\boldsymbol{\xi}} \\
&= \Psi(\underline{\mathbf{A}}) + \int \operatorname{grad} f(\underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle) \cdot \underline{\mathbf{a}}\langle\boldsymbol{\xi}\rangle\, dV_{\boldsymbol{\xi}}
\end{aligned}
$$

Therefore

$$
\Psi_{\underline{\mathbf{A}}}\langle\boldsymbol{\xi}\rangle = \operatorname{grad} f(\underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle).
$$

Now we have the tools in place to talk about elastic materials

Strain energy at $\mathbf{x}$ depends collectively on the deformation of the family of $\mathbf{x}$.



Undeformed family of $\mathbf{x}$

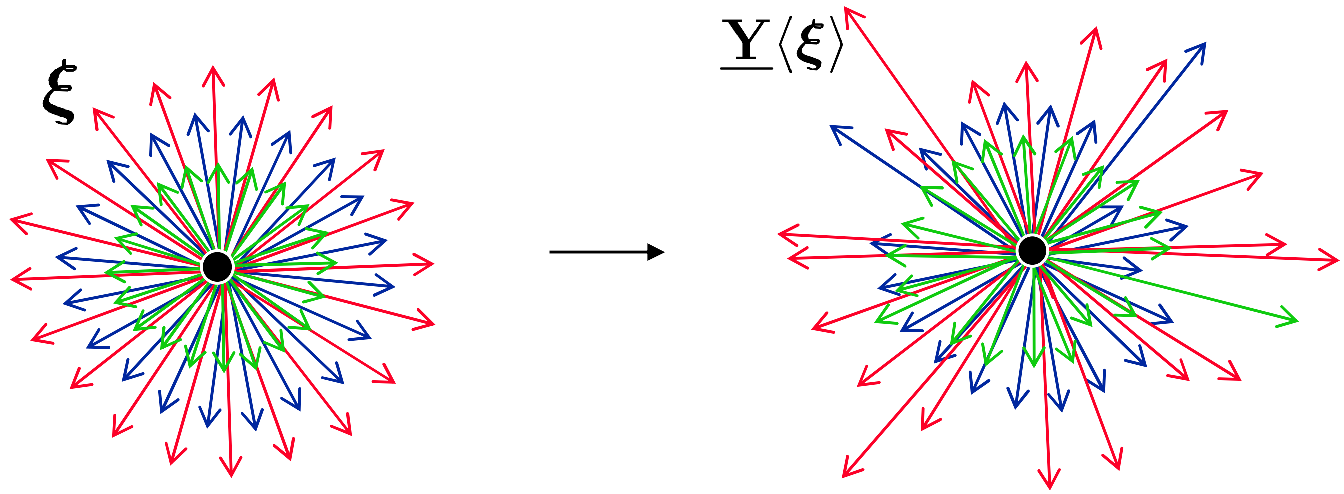
Deformed family of $\mathbf{x}$

Standard:	Peridynamic:
$W\left(\dfrac{\partial \mathbf{y}}{\partial \mathbf{x}}\right)$	$W(\underline{\mathbf{Y}})$

$\underline{\mathbf{Y}}$ is the *deformation state* defined by

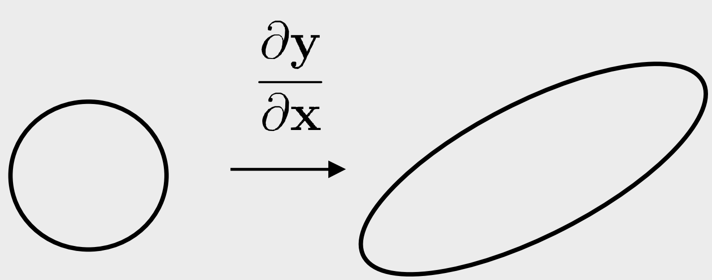
$$\underline{\mathbf{Y}}[\mathbf{x},t]\langle\mathbf{x}'-\mathbf{x}\rangle = \mathbf{y}(\mathbf{x}',t) - \mathbf{y}(\mathbf{x},t)$$

Deformation states can contain a lot of kinematical complexity



Undeformed bonds connected to $\mathbf{x}$

Deformed bonds connected to $\mathbf{x}$



Compare this with standard theory in which small spheres are mapped into ellipsoids

Force state is the work conjugate to the deformation state

- Suppose we perturb the deformed bond $\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle$ by a virtual displacement $\boldsymbol{\epsilon}$. The resulting change in $W(\mathbf{x})$ is

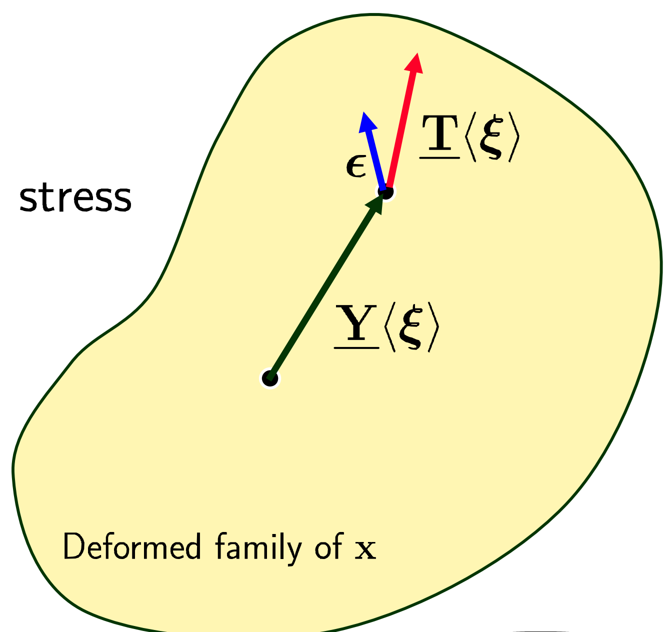
$$\Delta W = \underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle \cdot \boldsymbol{\epsilon}$$

  where $\underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle$ is a vector.

- The "force state" $\underline{\mathbf{T}}$ is the work conjugate to $\underline{\mathbf{Y}}$:

$$\dot{W} = \underline{\mathbf{T}} \bullet \dot{\underline{\mathbf{Y}}} = \int_{\mathcal{H}} \underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle \cdot \dot{\underline{\mathbf{Y}}}\langle\boldsymbol{\xi}\rangle \, dV_{\boldsymbol{\xi}}$$

- $\underline{\mathbf{T}}$ is the Frechet derivative of $W(\underline{\mathbf{Y}})$ – analogous to a stress tensor.



Displace just one bond $\boldsymbol{\xi}$

Potential energy and its first variation

- Total potential energy in $\mathcal{B}$:

$$\Phi = \int_{\mathcal{B}} \left(W(\underline{\mathbf{Y}}[\mathbf{x}]) - \mathbf{b}(\mathbf{x}) \cdot \mathbf{y}(\mathbf{x}) \right) dV_{\mathbf{x}}$$

- Take first variation.

$$\begin{aligned}
\delta\Phi &= \int_{\mathcal{B}} \left(W_{\underline{\mathbf{Y}}}[\mathbf{x}] \bullet \delta\underline{\mathbf{Y}}[\mathbf{x}] - \mathbf{b}(\mathbf{x}) \cdot \delta\mathbf{y}(\mathbf{x}) \right) dV_{\mathbf{x}} \\
&= \int_{\mathcal{B}} \left[\int_{\mathcal{B}} W_{\underline{\mathbf{Y}}}[\mathbf{x}]\langle \mathbf{x}' - \mathbf{x} \rangle \cdot \left(\delta\mathbf{y}(\mathbf{x}') - \delta\mathbf{y}(\mathbf{x}) \right) dV_{\mathbf{x}'} - \mathbf{b}(\mathbf{x}) \cdot \delta\mathbf{y}(\mathbf{x}) \right] dV_{\mathbf{x}} \\
&= \int_{\mathcal{B}} \left[\int_{\mathcal{B}} \left(W_{\underline{\mathbf{Y}}}[\mathbf{x}']\langle \mathbf{x} - \mathbf{x}' \rangle - W_{\underline{\mathbf{Y}}}[\mathbf{x}]\langle \mathbf{x}' - \mathbf{x} \rangle \right) dV_{\mathbf{x}'} - \mathbf{b}(\mathbf{x}) \right] \cdot \delta\mathbf{y}(\mathbf{x})\, dV_{\mathbf{x}}.
\end{aligned}$$

- Require $\delta\Phi = 0$ for all variations $\delta\mathbf{y}$. Euler-Lagrange equation is

$$\int_{\mathcal{H}} \left(W_{\underline{\mathbf{Y}}}[\mathbf{x}']\langle \mathbf{x} - \mathbf{x}' \rangle - W_{\underline{\mathbf{Y}}}[\mathbf{x}]\langle \mathbf{x}' - \mathbf{x} \rangle \right) dV_{\mathbf{x}'} - \mathbf{b}(\mathbf{x}) = \mathbf{0}$$

for all $\mathbf{x} \in \mathcal{B}$.

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Equilibrium equation

- Define the *force state* by

$$\underline{\mathbf{T}} = W_{\underline{\mathbf{Y}}}.$$

- Just showed that stationary potential energy implies the following *equilibrium equation*

$$\int_{\mathcal{H}} \Big(\underline{\mathbf{T}}[\mathbf{x}]\langle \mathbf{x}' - \mathbf{x} \rangle - \underline{\mathbf{T}}[\mathbf{x}']\langle \mathbf{x} - \mathbf{x}' \rangle \Big) \, dV_{\mathbf{x}'} + \mathbf{b}(\mathbf{x}) = \mathbf{0}$$

for all $\mathbf{x} \in \mathcal{B}$.

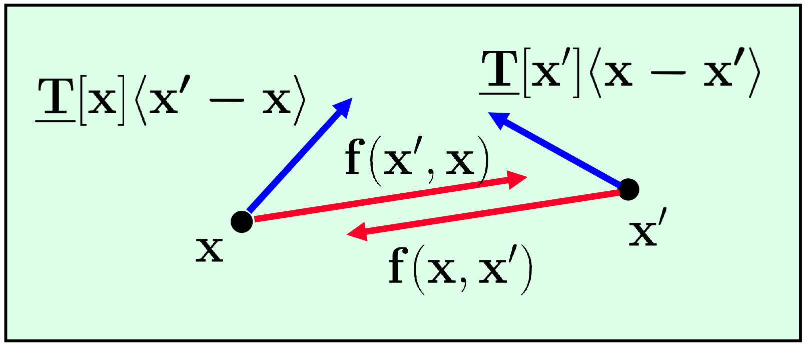
Bond force

- Equilibrium equation is

$$\int_{\mathcal{H}} \left(\underline{\mathbf{T}}[\mathbf{x}]\langle \mathbf{x}' - \mathbf{x} \rangle - \underline{\mathbf{T}}[\mathbf{x}']\langle \mathbf{x} - \mathbf{x}' \rangle \right) dV_{\mathbf{x}'} + \mathbf{b}(\mathbf{x}) = \mathbf{0}.$$

- Write this as:

$$\int_{\mathcal{H}} \mathbf{f}(\mathbf{x}', \mathbf{x}) \, dV_{\mathbf{x}'} + \mathbf{b}(\mathbf{x}) = \mathbf{0}.$$



- where the *bond force* is defined by

$$\mathbf{f}(\mathbf{x}', \mathbf{x}) = \underline{\mathbf{T}}[\mathbf{x}]\langle \mathbf{x}' - \mathbf{x} \rangle - \underline{\mathbf{T}}[\mathbf{x}']\langle \mathbf{x} - \mathbf{x}' \rangle$$

- Bond force is antisymmetric:

$$\mathbf{f}(\mathbf{x}, \mathbf{x}') = -\mathbf{f}(\mathbf{x}', \mathbf{x})$$

- In general the vector $\mathbf{f}(\mathbf{x}', \mathbf{x})$ is not parallel to the deformed bond $\underline{\mathbf{Y}}\langle \mathbf{x}' - \mathbf{x} \rangle$.

- $\mathbf{f}$ has dimensions of force/volume^2.

Principle of virtual work

- Equilibrium equation is

$$\int_{\mathcal{H}} \left(\underline{\mathbf{T}}[\mathbf{x}]\langle \mathbf{x}' - \mathbf{x} \rangle - \underline{\mathbf{T}}[\mathbf{x}']\langle \mathbf{x} - \mathbf{x}' \rangle \right) dV_{\mathbf{x}'} + \mathbf{b}(\mathbf{x}) = \mathbf{0}.$$

- Multiply by a virtual displacement field $\mathbf{w}$ and integrate:

$$\int_{\mathcal{B}} \int_{\mathcal{B}} \left(\underline{\mathbf{T}}[\mathbf{x}]\langle \mathbf{x}' - \mathbf{x} \rangle - \underline{\mathbf{T}}[\mathbf{x}']\langle \mathbf{x} - \mathbf{x}' \rangle \right) \cdot \mathbf{w}(\mathbf{x}) \; dV_{\mathbf{x}'} \; dV_{\mathbf{x}} + \int_{\mathcal{B}} \mathbf{b}(\mathbf{x}) \cdot \mathbf{w}(\mathbf{x}) \; dV_{\mathbf{x}} = \mathbf{0}$$

$$\int_{\mathcal{B}} \int_{\mathcal{B}} \underline{\mathbf{T}}[\mathbf{x}]\langle \mathbf{x}' - \mathbf{x} \rangle \cdot (\mathbf{w}(\mathbf{x}) - \mathbf{w}(\mathbf{x}')) \; dV_{\mathbf{x}'} \; dV_{\mathbf{x}} + \int_{\mathcal{B}} \mathbf{b}(\mathbf{x}) \cdot \mathbf{w}(\mathbf{x}) \; dV_{\mathbf{x}} = \mathbf{0}$$

- If we define $\underline{\mathbf{W}}$ to be the deformation state associated with $\mathbf{w}$

$$\underline{\mathbf{W}}[\mathbf{x}]\langle \mathbf{x}' - \mathbf{x} \rangle = \mathbf{w}(\mathbf{x}') - \mathbf{w}(\mathbf{x})$$

then the PVW is

$$\int_{\mathcal{B}} \underline{\mathbf{T}}[\mathbf{x}] \bullet \underline{\mathbf{W}}[\mathbf{x}] \; dV_{\mathbf{x}} - \int_{\mathcal{B}} \mathbf{b}(\mathbf{x}) \cdot \mathbf{w}(\mathbf{x}) \; dV_{\mathbf{x}} = \mathbf{0}.$$

Compare classical PVW
$\int (\sigma \cdot \nabla \mathbf{w} - \mathbf{b} \cdot \mathbf{w}) \; dV = \mathbf{0}$

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Peridynamic equation of motion

- Equilibrium equation:

$$\int_{\mathcal{H}} \mathbf{f}(\mathbf{x}', \mathbf{x})\, dV_{\mathbf{x}'} + \mathbf{b}(\mathbf{x}) = \mathbf{0}.$$

- where

$$\mathbf{f}(\mathbf{x}', \mathbf{x}) = \underline{\mathbf{T}}[\mathbf{x}]\langle \mathbf{x}' - \mathbf{x} \rangle - \underline{\mathbf{T}}[\mathbf{x}']\langle \mathbf{x} - \mathbf{x}' \rangle$$

- Now use d'Alembert's principle to get the equation of motion:

$$\rho(\mathbf{x})\ddot{\mathbf{y}}(\mathbf{x}, t) = \int_{\mathcal{H}} \mathbf{f}(\mathbf{x}', \mathbf{x}, t)\, dV_{\mathbf{x}'} + \mathbf{b}(\mathbf{x}, t)$$

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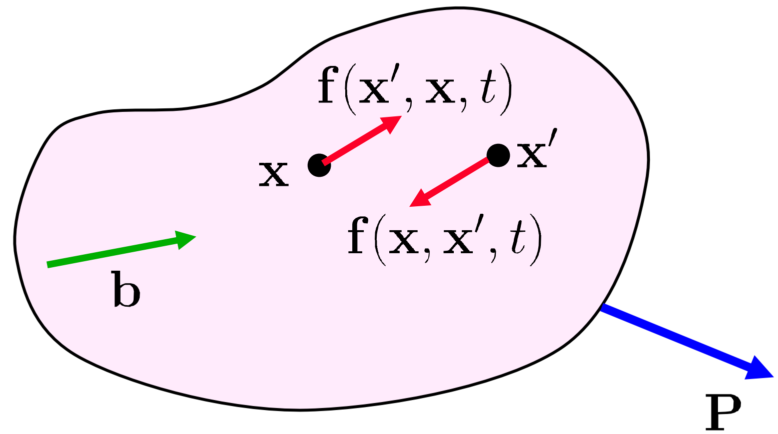
Balance of linear momentum

- Total linear momentum in the body:

$$\mathbf{P} = \int_{\mathcal{B}} \rho \dot{\mathbf{y}} \, dV_{\mathbf{x}}.$$



- Then

$$\begin{aligned} \dot{\mathbf{P}} &= \int_{\mathcal{B}} \rho \ddot{\mathbf{y}} \, dV_{\mathbf{x}} \\ &= \int_{\mathcal{B}} \left[\int_{\mathcal{B}} \mathbf{f}(\mathbf{x}', \mathbf{x}, t) \, dV'_{\mathbf{x}} + \mathbf{b} \right] \, dV_{\mathbf{x}} \end{aligned}$$

From equation of motion

- Recall $\mathbf{f}(\mathbf{x}', \mathbf{x}, t) = -\mathbf{f}(\mathbf{x}, \mathbf{x}', t)$, therefore

$$\dot{\mathbf{P}} = \int_{\mathcal{B}} \mathbf{b} \, dV_{\mathbf{x}}$$

- Rate of change of total momentum $=$ total applied force.

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Constitutive modeling

- A constitutive model relates the force state at a point $\mathbf{x}$ to the deformation state and any other variables:

$$\underline{\mathbf{T}} = \hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}}, \dot{\underline{\mathbf{Y}}}, \theta, \mathbf{x}, t, \ldots)$$

Rate dependence

Temperature

Heterogeneity

Explicit time dependence (e.g., ageing)

- Simple material:

$$\underline{\mathbf{T}} = \hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}}, \mathbf{x})$$

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Angular momentum balance

- Define the total angular momentum in the body by

$$\mathbf{A} = \int_{\mathcal{B}} \mathbf{y} \times \rho \dot{\mathbf{y}} \, dV_{\mathbf{x}}.$$

which says there are no "hidden" dofs that have angular momentum.

- Shorten the notation:

$$\mathbf{t} = \underline{\mathbf{T}}[\mathbf{x}, t]\langle \mathbf{x}' - \mathbf{x} \rangle, \qquad \mathbf{t}' = \underline{\mathbf{T}}[\mathbf{x}', t]\langle \mathbf{x} - \mathbf{x}' \rangle.$$

- Then

$$\begin{aligned} \dot{\mathbf{A}} &= \int_{\mathcal{B}} \mathbf{y} \times \rho \ddot{\mathbf{y}} \, dV_{\mathbf{x}} \\ &= \int_{\mathcal{B}} \mathbf{y} \times \left[\int_{\mathcal{B}} (\mathbf{t} - \mathbf{t}') \, dV_{\mathbf{x}'} + \mathbf{b} \right] \, dV_{\mathbf{x}} \end{aligned}$$

From equation of motion

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Angular momentum balance: Nonpolar materials

$$
\begin{aligned}
\dot{\mathbf{A}} &= \int_{\mathcal{B}}\int_{\mathcal{B}} \mathbf{y} \times (\mathbf{t} - \mathbf{t}')\, dV_{\mathbf{x}'}\, dV_{\mathbf{x}} + \int_{\mathcal{B}} \mathbf{y} \times \mathbf{b}\, dV_{\mathbf{x}} \\
&= \int_{\mathcal{B}}\int_{\mathcal{B}} (\mathbf{y} - \mathbf{y}') \times \mathbf{t}\, dV_{\mathbf{x}'}\, dV_{\mathbf{x}} + \int_{\mathcal{B}} \mathbf{y} \times \mathbf{b}\, dV_{\mathbf{x}} \\
&= -\int_{\mathcal{B}}\int_{\mathcal{H}} \underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle \times \underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle\, dV_{\boldsymbol{\xi}}\, dV_{\mathbf{x}} + \int_{\mathcal{B}} \mathbf{y} \times \mathbf{b}\, dV_{\mathbf{x}}
\end{aligned}
$$

Suppose the constitutive model is *nonpolar*:

$$
\int_{\mathcal{H}} \underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle \times \underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle\, dV_{\boldsymbol{\xi}} = \mathbf{0}
$$

for all $\underline{\mathbf{Y}}$. Then

$$
\dot{\mathbf{A}} = \int_{\mathcal{B}} \mathbf{y} \times \mathbf{b}\, dV_{\mathbf{x}}
$$

which says there are no “hidden” moments.

Nonpolar materials

- Nonpolarity:

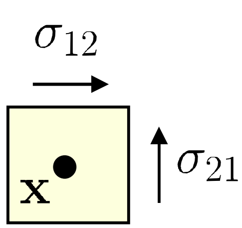
$$\int_{\mathcal{H}} \underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle \times \underline{\hat{\mathbf{T}}}(\underline{\mathbf{Y}})\langle\boldsymbol{\xi}\rangle \, dV_{\boldsymbol{\xi}} = \mathbf{0} \qquad \forall \underline{\mathbf{Y}}$$

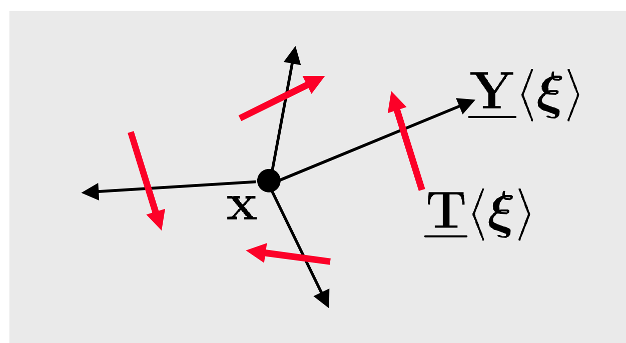
  implies the global balance of angular momentum.

- Converse can be proved too (global balance of angular momentum implies material is nonpolar).

- We will adopt nonpolarity as a constitutive restriction.

- "No net moment on a point due to its own force state."





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Ordinary and nonordinary

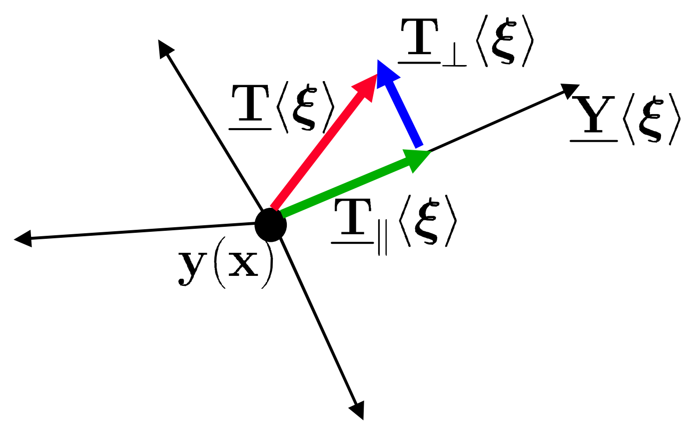
- Any force state can be decomposed into parts that are parallel and orthogonal to the deformed bonds:

$$\underline{\mathbf{T}} = \underline{\mathbf{T}}_{\parallel} + \underline{\mathbf{T}}_{\perp}$$

where

$$\underline{\mathbf{T}}_{\parallel}\langle\boldsymbol{\xi}\rangle = (\underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle \cdot \underline{\mathbf{M}}\langle\boldsymbol{\xi}\rangle)\underline{\mathbf{M}}\langle\boldsymbol{\xi}\rangle$$

$$\underline{\mathbf{M}}\langle\boldsymbol{\xi}\rangle = \frac{\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle}{|\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle|}.$$



- If

$$\underline{\mathbf{T}}_{\perp} = \underline{\mathbf{0}} \qquad \forall \underline{\mathbf{Y}}$$

then the material is *ordinary*, otherwise *nonordinary*.

Elastic materials: objectivity implies nonpolarity

- Objectivity: for any proper orthogonal tensor $\mathbf{Q}$,

$$W(\mathbf{Q}\underline{\mathbf{Y}}) = W(\underline{\mathbf{Y}}).$$

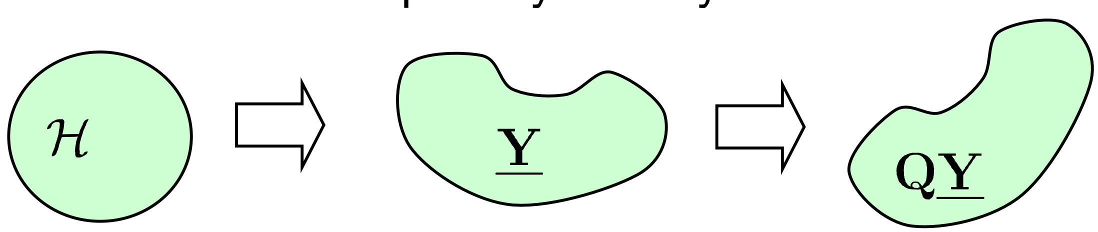
  i.e., energy doesn't change if you rigidly rotate the family after deforming it.

- Can show that any objective, elastic material is nonpolar:

$$\int_{\mathcal{H}} \underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle \times W_{\underline{\mathbf{Y}}}\langle\boldsymbol{\xi}\rangle \, dV_{\boldsymbol{\xi}} = \mathbf{0}.$$

  Details: see Silling, "Linearized theory of peridynamic states," J. Elast. (2010).

- Result is important because usually objectivity is much easier to determine than nonpolarity directly.



Energy balance

- Recall that for an elastic material, since $\underline{\mathbf{T}} = W_{\underline{\mathbf{Y}}}$,

$$W(\underline{\mathbf{Y}} + \delta\underline{\mathbf{Y}}) = W(\underline{\mathbf{Y}}) + \underline{\mathbf{T}} \bullet \delta\underline{\mathbf{Y}} + o(||\delta\underline{\mathbf{Y}}||)$$

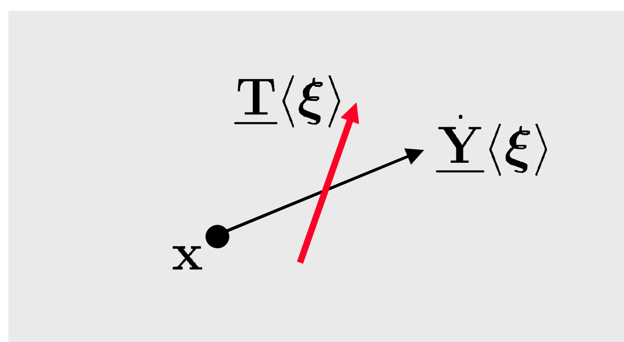
therefore

$$\dot{W} = \underline{\mathbf{T}} \bullet \dot{\underline{\mathbf{Y}}}$$

Compare stress power: $\dot{W} = \boldsymbol{\sigma} \cdot \dot{\mathbf{F}}$.

i.e.,

$$\dot{W}(\mathbf{x}, t) = \int_{\mathcal{H}} \underline{\mathbf{T}}[\mathbf{x}, t]\langle \mathbf{x}' - \mathbf{x} \rangle \cdot (\dot{\mathbf{y}}(\mathbf{x}', t) - \dot{\mathbf{y}}(\mathbf{x}, t))\, dV_{\mathbf{x}'}$$



Energy balance, ctd.

- For more general materials, the first law of thermodynamics is

$$\dot{\varepsilon} = \underline{\mathbf{T}} \bullet \underline{\dot{\mathbf{Y}}} + h + r$$

where ε=internal energy density, h=net heat transport rate to $\mathbf{x}$ per unit volume, r = heat source rate.

- This applies to any heat transport law, e.g.

$$h = K\nabla^2\theta \qquad \text{Fourier's law, local}$$

or

$$h = \int_{\mathcal{B}} K(\mathbf{x}' - \mathbf{x})(\theta(\mathbf{x}', t) - \theta(\mathbf{x}, t))\, dV_{\mathbf{x}'} \qquad \text{nonlocal}$$

Free energy and 2nd law of thermodynamics

(joint work with Rich Lehoucq, thanks also to Eliot Fried)

- Since we're now dealing with temperature, have to include it in the internal energy:

$$\varepsilon(\underline{\mathbf{Y}}, \theta).$$

- Now try to find $\underline{\mathbf{T}}$ from ε. Define the *free energy* by

$$\psi = \varepsilon - \theta\eta$$

  where η=entropy.

- Thus

$$\dot{\psi} = \dot{\varepsilon} - \dot{\theta}\eta - \theta\dot{\eta}.$$

- Hence from 1st law

$$\dot{\psi} = \underline{\mathbf{T}} \bullet \underline{\dot{\mathbf{Y}}} + h + r - \dot{\theta}\eta - \theta\dot{\eta}.$$

- Second law (Clausius inequality):

$$\theta\dot{\eta} \geq h + r.$$

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Free energy and the force state

- From last two equations,

$$\underline{\mathbf{T}} \bullet \dot{\underline{\mathbf{Y}}} - \dot{\theta}\eta - \dot{\psi} \geq 0.$$

- Assume $\psi = \psi(\underline{\mathbf{Y}}, \theta)$. Therefore

$$\underline{\mathbf{T}} \bullet \dot{\underline{\mathbf{Y}}} - \dot{\theta}\eta - (\psi_{\underline{\mathbf{Y}}} \bullet \dot{\underline{\mathbf{Y}}} + \psi_\theta \dot{\theta}) \geq 0.$$

- Group terms:

$$(\underline{\mathbf{T}} - \psi_{\underline{\mathbf{Y}}}) \bullet \dot{\underline{\mathbf{Y}}} - (\eta + \psi_\theta)\dot{\theta} \geq 0.$$

- Since $\underline{\mathbf{Y}}$ and θ can (in principle) be varied independently, conclude (from Coleman-Noll argument) that

$$\underline{\mathbf{T}} = \psi_{\underline{\mathbf{Y}}} \qquad \text{and} \qquad \eta = -\psi_\theta.$$

- Special case: if ψ is independent of θ, get an elastic material with $W = \psi$.

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Rate dependent materials

- If we allow rate dependence in the model,

$$\psi = \psi(\underline{\mathbf{Y}}, \underline{\dot{\mathbf{Y}}}, \theta)$$

can show that the force state can be decomposed into *equilibrium* and *dissipative* parts:

$$\underline{\mathbf{T}} = \underline{\mathbf{T}}^e(\underline{\mathbf{Y}}, \theta) + \underline{\mathbf{T}}^d(\underline{\mathbf{Y}}, \underline{\dot{\mathbf{Y}}}, \theta)$$

where

$$\underline{\mathbf{T}}^e = \psi_{\underline{\mathbf{Y}}} \qquad \text{and} \qquad \underline{\mathbf{T}}^d \bullet \underline{\dot{\mathbf{Y}}} \geq 0.$$

Energy dissipation

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Material modeling: Bond-based materials

- The simplest assumption is that all the bonds are independent.

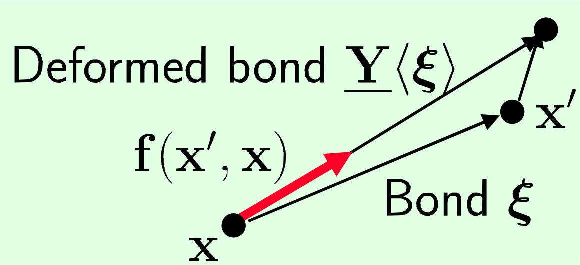
$$W(\underline{\mathbf{Y}}) = \int_{\mathcal{H}} \psi(\underline{e}\langle\boldsymbol{\xi}\rangle, \boldsymbol{\xi})\, dV_{\boldsymbol{\xi}}, \qquad \underline{e}\langle\boldsymbol{\xi}\rangle = |\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle| - |\boldsymbol{\xi}|$$

$$\underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle = \psi'(\underline{e}\langle\boldsymbol{\xi}\rangle, \boldsymbol{\xi})\mathbf{M}, \qquad \mathbf{M} = \frac{\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle}{|\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle|}$$

- Equation of motion simplifies to

$$\rho\ddot{\mathbf{y}}(\mathbf{x}, t) = \int_{\mathcal{H}} \mathbf{f}(\mathbf{x}', \mathbf{x})\, dV_{\mathbf{x}} + \mathbf{b}(\mathbf{x}, t),$$

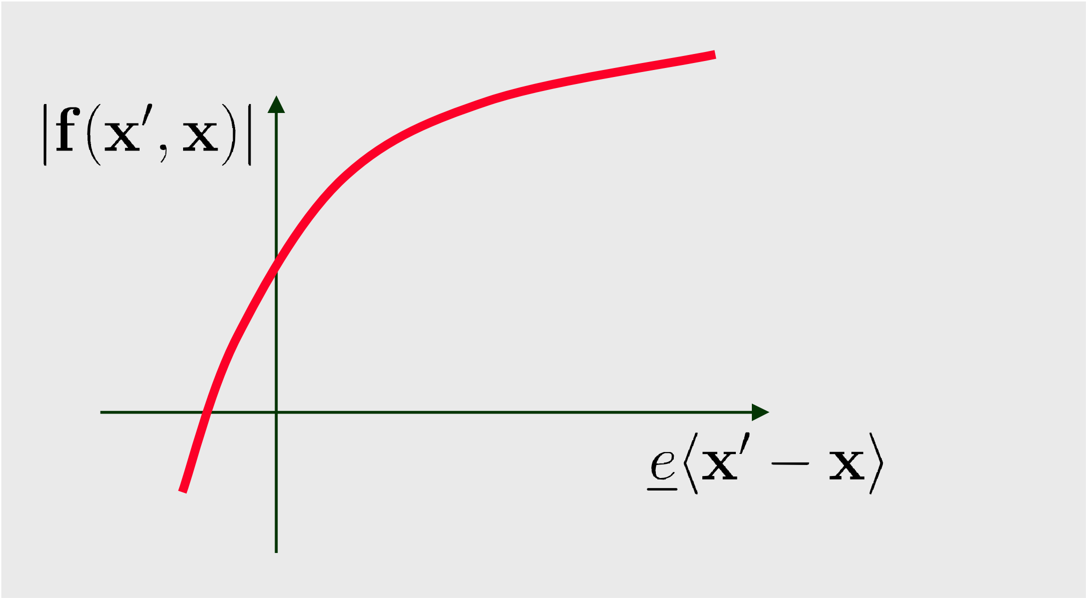
$$\mathbf{f}(\mathbf{x}', \mathbf{x}) = \big(\psi'(\underline{e}\langle\boldsymbol{\xi}\rangle, \boldsymbol{\xi}) + \psi'(\underline{e}\langle\boldsymbol{\xi}\rangle, -\boldsymbol{\xi})\big)\mathbf{M}$$



Bond extension $= \underline{e}\langle\boldsymbol{\xi}\rangle = |\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle| - |\boldsymbol{\xi}|$

Material modeling: Bond–based materials, ctd.

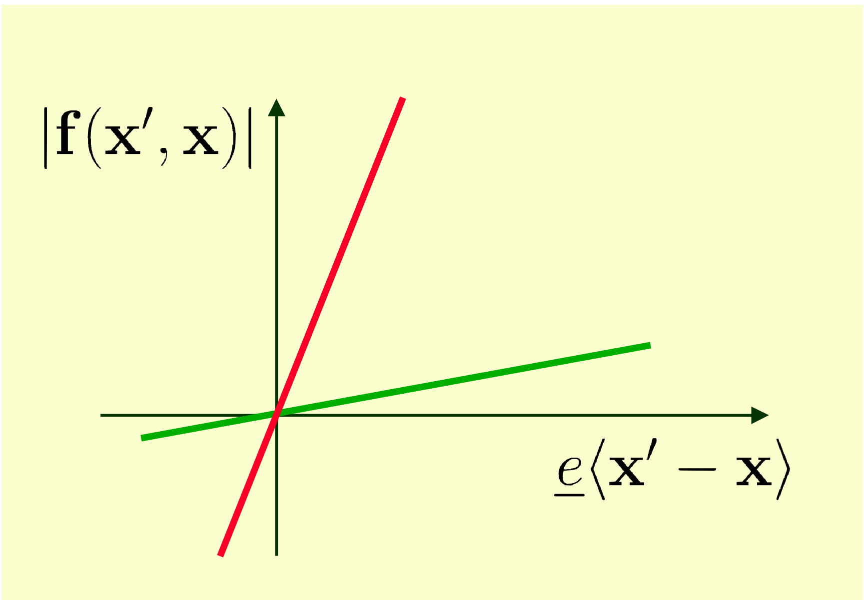
- The body is a network of independent, nonlinear springs.
- Material response is described by a graph of bond force vs. bond extension.
- If the material is isotropic, the Poisson ratio $= 1/4$ (!).

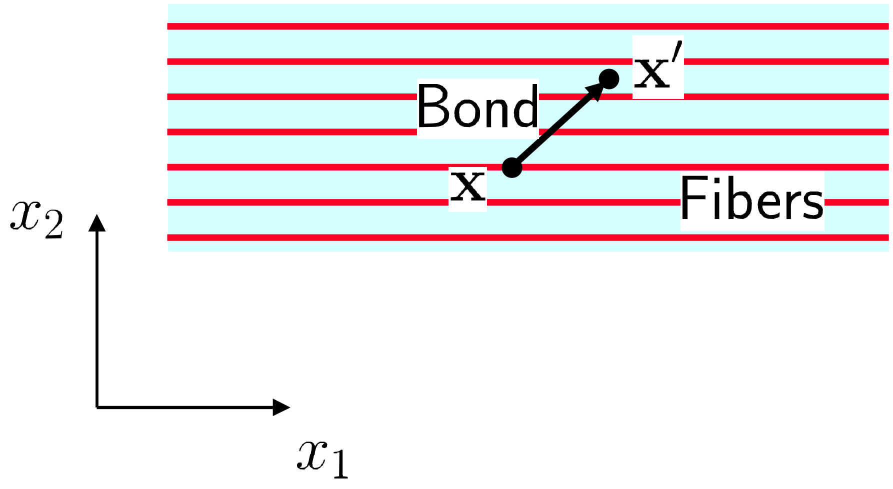


Material modeling: Bond-based materials, ctd.

- Special case: fiber reinforced composite lamina.
- Bonds in the fiber direction are stiffer than the others.

$$\mathbf{f}(\mathbf{x}', \mathbf{x}) = \big(c_1 + c_2 \Delta(x_2' - x_2)\big)\underline{e}\langle \mathbf{x}' - \mathbf{x} \rangle \mathbf{M}$$





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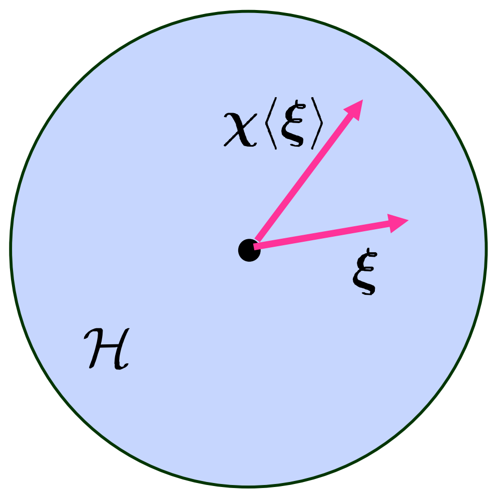
Material modeling: Bond–pair materials

- Suppose every bond $\boldsymbol{\xi}$ has a friend $\boldsymbol{\eta} = \boldsymbol{\chi}(\boldsymbol{\xi})$. The material responds to the deformation of pairs of bonds.

$$W(\underline{\mathbf{Y}}) = \int_{\mathcal{H}} \psi(\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle, \underline{\mathbf{Y}}\langle\boldsymbol{\eta}\rangle, \boldsymbol{\xi}, \boldsymbol{\eta}) \, dV_{\boldsymbol{\xi}}$$

where ψ is a function of four vectors:

$$\psi(\mathbf{p}, \mathbf{q}, \boldsymbol{\xi}, \boldsymbol{\eta}).$$



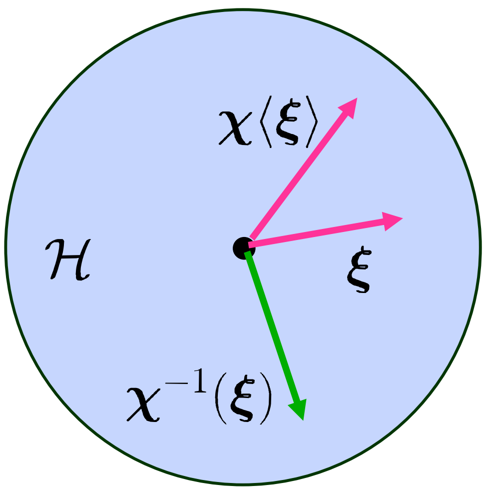
Material modeling: Bond–pair materials, ctd.

- Fréchet derivative yields

$$\underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle = \psi_{\mathbf{p}}\big(\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle, \underline{\mathbf{Y}}\langle\boldsymbol{\chi}(\boldsymbol{\xi})\rangle, \boldsymbol{\xi}, \chi(\boldsymbol{\xi})\big) + \psi_{\mathbf{q}}\big(\underline{\mathbf{Y}}\langle\boldsymbol{\chi}^{-1}(\boldsymbol{\xi})\rangle, \underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle, \boldsymbol{\chi}^{-1}(\boldsymbol{\xi}), \boldsymbol{\xi}\big) J^{-1}$$

where

$$J = \big|\det \operatorname{grad} \boldsymbol{\chi}\big|$$



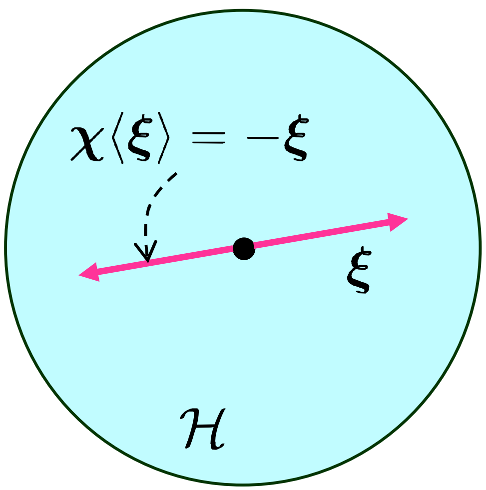
Material modeling: Bond–pair materials, ctd.

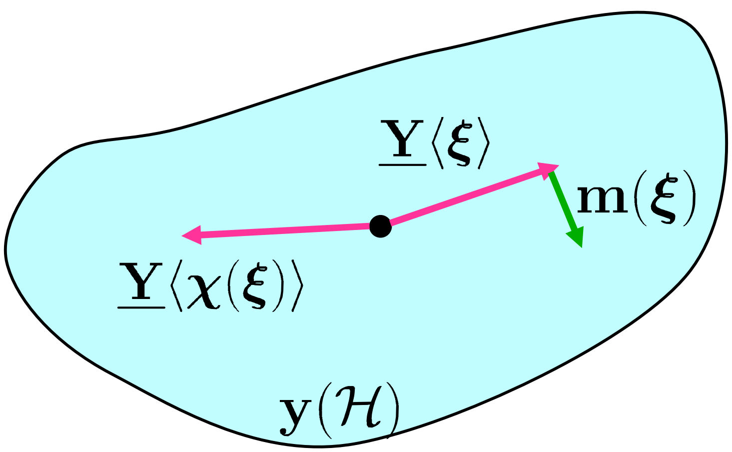
- Specific case:

$$\boldsymbol{\chi}(\boldsymbol{\xi}) = -\boldsymbol{\xi}$$

$$\psi(\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle, \underline{\mathbf{Y}}\langle\boldsymbol{\eta}\rangle, \boldsymbol{\xi}, \boldsymbol{\eta}) = \frac{c}{4}(\theta - \pi)^2, \qquad \theta = \cos^{-1}\frac{\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle \cdot \underline{\mathbf{Y}}\langle\boldsymbol{\eta}\rangle}{|\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle|\;|\underline{\mathbf{Y}}\langle\boldsymbol{\eta}\rangle|}$$

$$\underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle = \frac{c(\pi - \theta)}{|\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle|}\mathbf{m}(\boldsymbol{\xi})$$





$\mathbf{m}(\boldsymbol{\xi})$= unit vector $\perp$ to $\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle$

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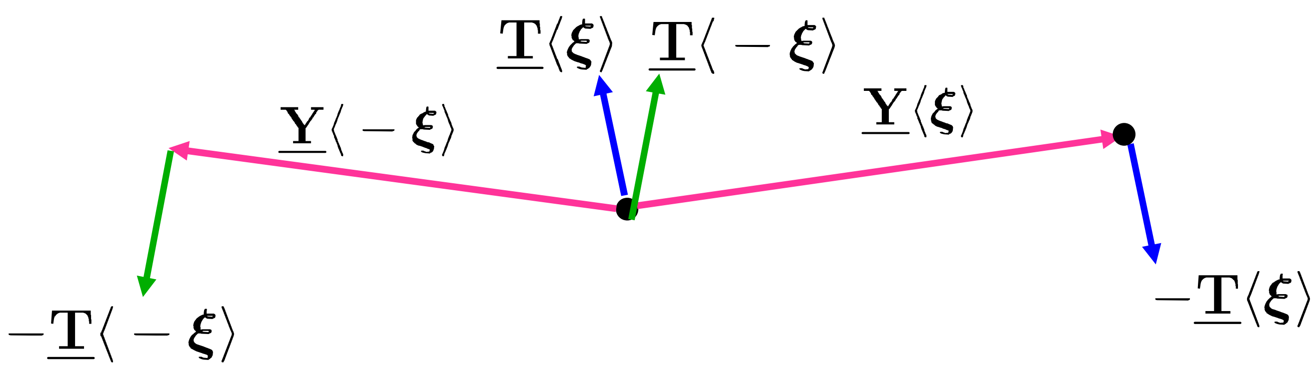
Material modeling: Bond–pair materials, ctd.

Fascinating facts:

- This material does not respond at all to homogeneous deformation.
- It provides a consistent way to model bending of a one-dimensional beam.
- The standard model for a beam involves introducing a different theory from the continuum theory.



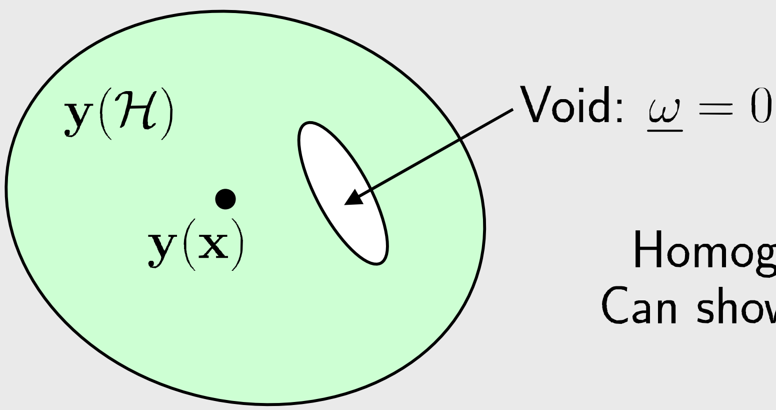
Using a classical stress-strain material model in peridynamics

- Suppose we are given a model for the Piola stress: $\boldsymbol{\sigma}(\mathbf{F})$, where $\mathbf{F} = \nabla \mathbf{y}$.
- Want to use this somehow to get a force state. Define an approximate $\mathbf{F}$ by

$$\bar{\mathbf{F}} = \left[\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle \otimes \boldsymbol{\xi} \, dV_{\boldsymbol{\xi}} \right] \mathbf{K}^{-1}$$

where $\underline{\omega}$ is a given *influence function* and $\mathbf{K}$ is the *shape tensor*:

$$\mathbf{K} = \int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle \boldsymbol{\xi} \otimes \boldsymbol{\xi} \, dV_{\boldsymbol{\xi}}$$



Homogeneous deformation:
Can show $\bar{\mathbf{F}} = \mathbf{F}$ even near boundaries.

Using a classical stress-strain material model in peridynamics, ctd.

- Set

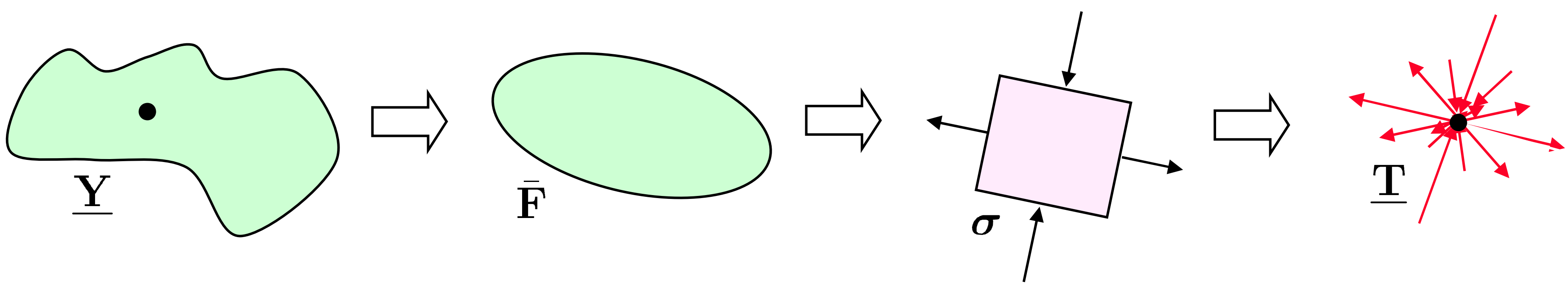
$$\underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle = \underline{\omega}\langle\boldsymbol{\xi}\rangle\boldsymbol{\sigma}(\bar{\mathbf{F}})\mathbf{K}^{-1}\boldsymbol{\xi} \qquad \forall\boldsymbol{\xi}$$

- Can show that if $\boldsymbol{\sigma} = \partial W/\partial\mathbf{F}$, then $\underline{\mathbf{T}} = W_{\underline{\mathbf{Y}}}$.
- Can also show that if the Cauchy stress tensor is symmetric, i.e.,

$$\boldsymbol{\tau}^T = \boldsymbol{\tau} \qquad \text{where} \qquad \boldsymbol{\tau} = \frac{\boldsymbol{\sigma}(\mathbf{F})\mathbf{F}^T}{\det\mathbf{F}}$$

then the peridynamic material is nonpolar:

$$\int_{\mathcal{H}} \underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle \times \underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle \; dV_{\boldsymbol{\xi}} = \mathbf{0} \qquad \forall\underline{\mathbf{Y}}.$$



Fluids

- Define a nonlocal dilatation based on the mean bond extension:

$$\vartheta = \frac{3}{m} \underline{\omega x} \bullet \underline{e}$$

where

$$\underline{x}\langle \boldsymbol{\xi} \rangle = |\boldsymbol{\xi}|, \qquad m = \underline{\omega x} \bullet \underline{x}, \qquad \underline{e}\langle \boldsymbol{\xi} \rangle = |\underline{\mathbf{Y}}\langle \boldsymbol{\xi} \rangle| - \underline{x}\langle \boldsymbol{\xi} \rangle.$$

- Writing this out in detail:

$$\vartheta = \frac{3}{m} \int_{\mathcal{H}} \underline{\omega}\langle \boldsymbol{\xi} \rangle |\boldsymbol{\xi}| \big(|\underline{\mathbf{Y}}\langle \boldsymbol{\xi} \rangle| - |\boldsymbol{\xi}| \big) \, dV_{\boldsymbol{\xi}}.$$

- Constitutive model: $W(\vartheta)$.

Fluids, ctd.

- Nonlocal dilatation:

$$\vartheta(\underline{\mathbf{Y}}) = \frac{3}{m}\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle|\boldsymbol{\xi}|\big(|\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle| - |\boldsymbol{\xi}|\big)\, dV_{\boldsymbol{\xi}}.$$

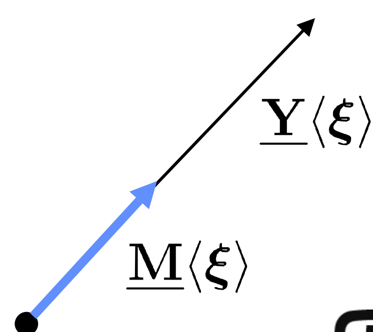
- Fréchet derivative of dilatation: observe

$$\begin{aligned}
\vartheta(\underline{\mathbf{Y}}+\delta\underline{\mathbf{Y}}) &= \frac{3}{m}\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle|\boldsymbol{\xi}|\big(|\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle + \delta\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle| - |\boldsymbol{\xi}|\big)\, dV_{\boldsymbol{\xi}} \\
&= \frac{3}{m}\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle|\boldsymbol{\xi}|\left(|\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle| + \frac{\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle}{|\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle|}\cdot\delta\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle - |\boldsymbol{\xi}|\right) dV_{\boldsymbol{\xi}} \\
&= \vartheta(\underline{\mathbf{Y}}) + \frac{3}{m}\int_{\mathcal{H}} \underline{\omega}\langle\boldsymbol{\xi}\rangle|\boldsymbol{\xi}|\left(\frac{\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle}{|\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle|}\cdot\delta\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle\right) dV_{\boldsymbol{\xi}}
\end{aligned}$$

hence

$$\vartheta_{\underline{\mathbf{Y}}} = \frac{3}{m}\underline{\omega x \mathbf{M}} \qquad \text{where} \qquad \underline{\mathbf{M}} = \frac{\underline{\mathbf{Y}}}{|\underline{\mathbf{Y}}|}.$$

$\underline{\mathbf{M}}\langle\boldsymbol{\xi}\rangle$ is the deformed bond direction

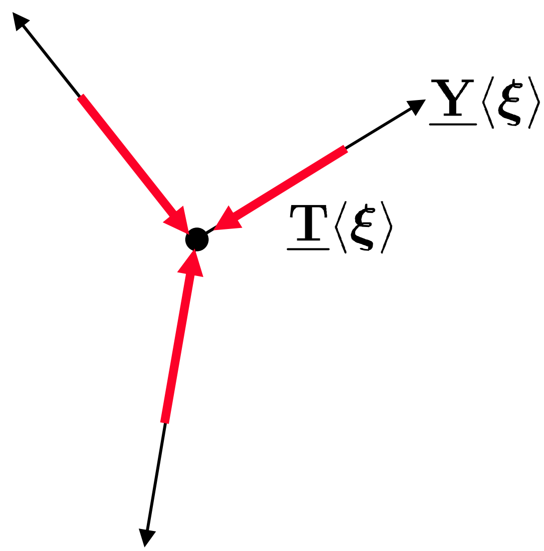


Fluids, ctd.

- Constitutive model is $W(\vartheta)$.
- Now can write down the force state: chain rule implies

$$\hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}}) = W_{\underline{\mathbf{Y}}}(\vartheta(\underline{\mathbf{Y}})) = \frac{dW}{d\vartheta}(\vartheta(\underline{\mathbf{Y}}))\vartheta_{\underline{\mathbf{Y}}} = \frac{dW}{d\vartheta}(\vartheta(\underline{\mathbf{Y}}))\ \frac{3\underline{\omega x \mathbf{M}}}{m}.$$

- Nonlocal pressure $= -dW/d\vartheta$.
- Bond forces are parallel to the deformed bonds (material is ordinary).



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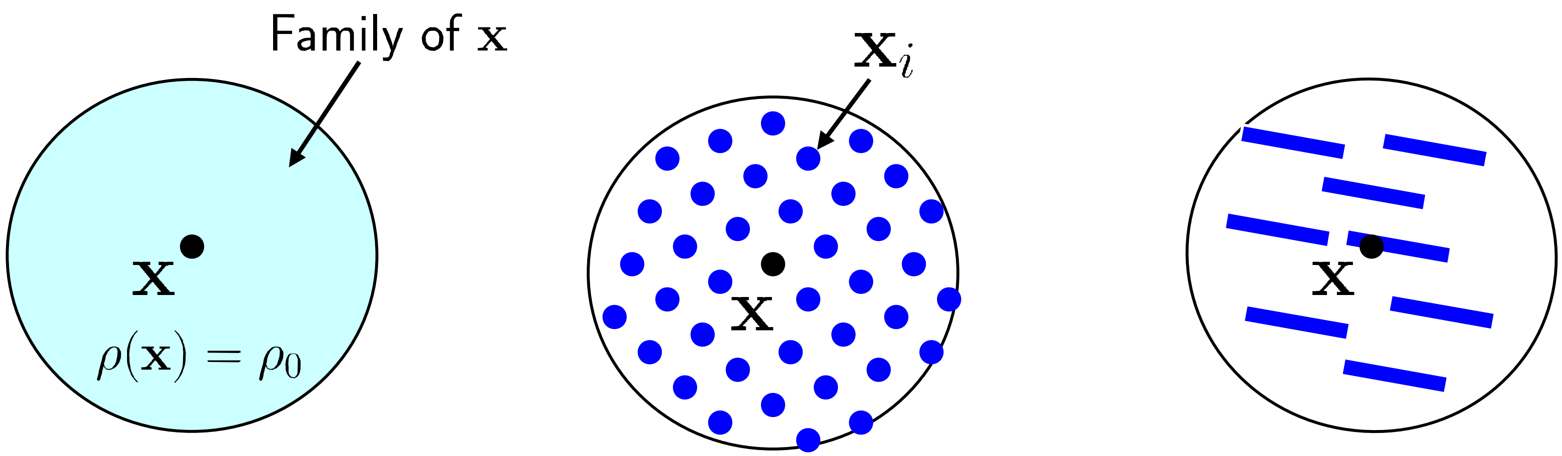
Material modeling: Discrete particles

- The family of $\mathbf{x}$ could be either continuous or a collection of point masses or other objects.



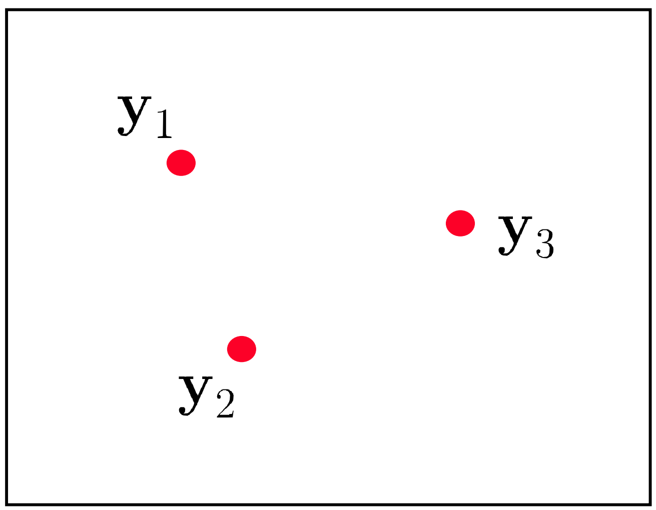
$$\rho(\mathbf{x}) = m \sum_i \Delta(\mathbf{x} - \mathbf{x}_i)$$

Δ = 3D Dirac delta function

Material modeling:
Discrete particles, ctd.

- Consider a set of atoms that interact through an $N-$body potential:
$$U(\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_N),$$
$\mathbf{y}_1, \dots, \mathbf{y}_N$ = deformed positions, $\mathbf{x}_1, \dots, \mathbf{x}_N$ = reference positions.

- This can be represented exactly as a peridynamic body.



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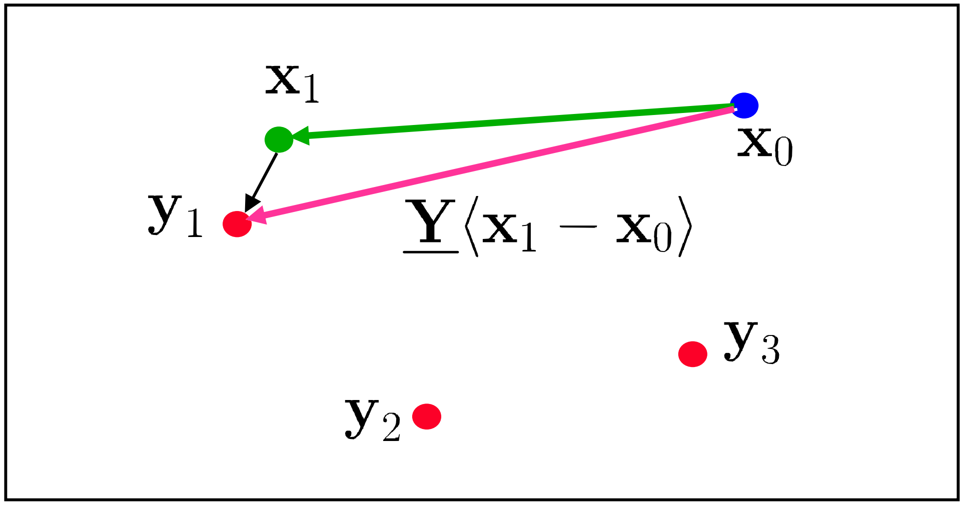
Material modeling: Discrete particles, ctd.

Define a peridynamic body by:

$$\hat{W}(\underline{\mathbf{Y}}, \mathbf{x}) = \Delta(\mathbf{x}-\mathbf{x}_0)U(\underline{\mathbf{Y}}\langle\mathbf{x}_1-\mathbf{x}_0\rangle, \underline{\mathbf{Y}}\langle\mathbf{x}_2-\mathbf{x}_0\rangle, \ldots, \underline{\mathbf{Y}}\langle\mathbf{x}_N-\mathbf{x}_0\rangle),$$

$$\rho(\mathbf{x}) = \sum_i \Delta(\mathbf{x}-\mathbf{x}_i)M_i$$



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Material modeling: Discrete particles, ctd.

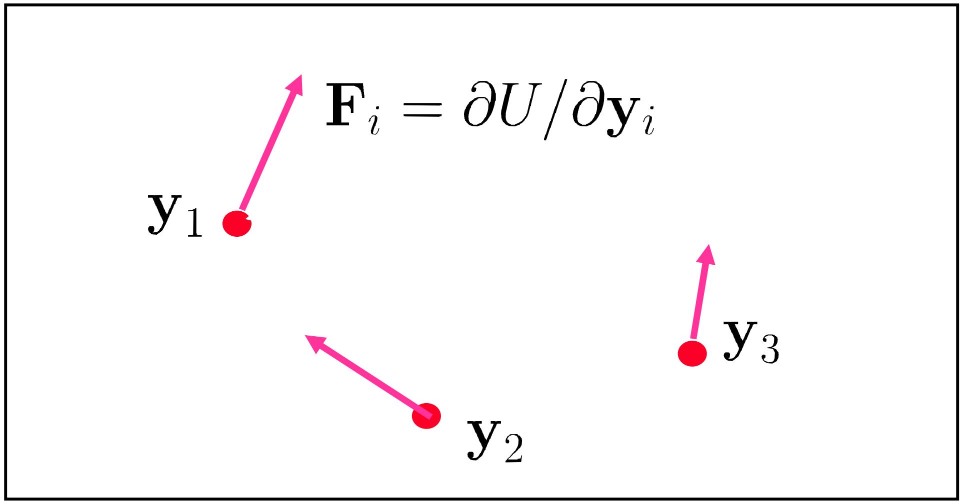
- After evaluating the Frechet derivative $\underline{\mathbf{T}}$, find

$$\underline{\mathbf{T}}[\mathbf{x}]\langle\boldsymbol{\xi}\rangle = \Delta(\mathbf{x}-\mathbf{x}_0)\sum_i \frac{\partial U}{\partial \mathbf{y}_i}\Delta\big(\boldsymbol{\xi}-(\mathbf{x}_i-\mathbf{x}_0)\big)$$

- Equation of motion reduces to

$$M_i\ddot{\mathbf{y}}(\mathbf{x}_i,t) = -\frac{\partial U}{\partial \mathbf{y}_i}, \qquad i=1,\ldots,N$$

- Have represented a multibody potential exactly within a continuum model.



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Linearization of a material model

- Small displacement field $\mathbf{u}$ superposed on a (possibly) large deformation $\mathbf{y}^0$:

$$\hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}}^0 + \underline{\mathbf{U}}) = \hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}}^0) + \underline{\mathbb{K}} \bullet \underline{\mathbf{U}} + o(||\underline{\mathbf{U}}||)$$

where

$$\underline{\mathbf{Y}}^0\langle \mathbf{x}' - \mathbf{x} \rangle = \mathbf{y}(\mathbf{x}') - \mathbf{y}(\mathbf{x})$$

$$\underline{\mathbf{U}}\langle \mathbf{x}' - \mathbf{x} \rangle = \mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})$$

$$\underline{\mathbb{K}} = \hat{\underline{\mathbf{T}}}_{\underline{\mathbf{Y}}}(\underline{\mathbf{Y}}^0)$$

- $\underline{\mathbb{K}}\langle \boldsymbol{\xi}, \boldsymbol{\zeta} \rangle$ is a *double state* (tensor valued function of two bonds):

$$(\underline{\mathbb{K}} \bullet \underline{\mathbf{U}})\langle \boldsymbol{\xi} \rangle = \int_{\mathcal{H}} \underline{\mathbb{K}}\langle \boldsymbol{\xi}, \boldsymbol{\zeta} \rangle \, \underline{\mathbf{U}}\langle \boldsymbol{\zeta} \rangle \, dV_{\zeta}$$

Linearization of an elastic material model

- If $\underline{\hat{\mathbf{T}}}$ is elastic,
$$\underline{\mathbb{K}} = \underline{\hat{\mathbf{T}}}_{\underline{\mathbf{Y}}}(\underline{\mathbf{Y}}^0) = W_{\underline{\mathbf{Y}\mathbf{Y}}}(\underline{\mathbf{Y}}^0)$$
i.e., $\underline{\mathbb{K}}$ is the second Fréchet derivative of W.
- Can show that for a linearized elastic material,
$$\underline{\mathbb{K}}\langle \boldsymbol{\zeta}, \boldsymbol{\xi} \rangle = \underline{\mathbb{K}}^T \langle \boldsymbol{\xi}, \boldsymbol{\zeta} \rangle \qquad \forall \boldsymbol{\xi}, \boldsymbol{\zeta}$$
- Converse is also true.
- $\underline{\mathbb{K}}$ is called the *modulus state*, similar to 4th order elasticity tensor.

Equation of motion for a linearized material

- If $\mathbf{y}^0$ is equilibrated,

$$\begin{aligned}
\rho\ddot{\mathbf{u}}(\mathbf{x}) &= \int \left(\underline{\mathbf{T}}[\mathbf{x}]\langle\mathbf{p}-\mathbf{x}\rangle - \underline{\mathbf{T}}[\mathbf{p}]\langle\mathbf{x}-\mathbf{p}\rangle\right) dV_{\mathbf{p}} + \mathbf{b}(\mathbf{x}) \\
&= \int \Big((\underline{\mathbb{K}}[\mathbf{x}] \bullet \underline{\mathbf{U}}[\mathbf{x}])\langle\mathbf{p}-\mathbf{x}\rangle - (\underline{\mathbb{K}}[\mathbf{p}] \bullet \underline{\mathbf{U}}[\mathbf{p}])\langle\mathbf{x}-\mathbf{p}\rangle\Big) dV_{\mathbf{p}} + \mathbf{b}(\mathbf{x}) \\
&= \int\int \Big(\underline{\mathbb{K}}[\mathbf{x}]\langle\mathbf{p}-\mathbf{x},\mathbf{q}-\mathbf{x}\rangle(\mathbf{u}(\mathbf{q})-\mathbf{u}(\mathbf{x})) - \underline{\mathbb{K}}[\mathbf{p}]\langle\mathbf{x}-\mathbf{p},\mathbf{q}-\mathbf{p}\rangle(\mathbf{u}(\mathbf{q})-\mathbf{u}(\mathbf{p}))\Big) dV_{\mathbf{q}}\, dV_{\mathbf{p}} + \mathbf{b}(\mathbf{x}) \\
&= \int \mathbf{C}(\mathbf{x},\mathbf{q})(\mathbf{u}(\mathbf{q})-\mathbf{u}(\mathbf{x}))\, dV_{\mathbf{q}} + \mathbf{b}(\mathbf{x})
\end{aligned}$$

where

$$\mathbf{C}(\mathbf{x},\mathbf{q}) = \int \Big(\underline{\mathbb{K}}[\mathbf{x}]\langle\mathbf{p}-\mathbf{x},\mathbf{q}-\mathbf{x}\rangle - \underline{\mathbb{K}}[\mathbf{p}]\langle\mathbf{x}-\mathbf{p},\mathbf{q}-\mathbf{p}\rangle + \underline{\mathbb{K}}[\mathbf{q}]\langle\mathbf{x}-\mathbf{q},\mathbf{p}-\mathbf{q}\rangle\Big) dV_{\mathbf{p}}$$

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Equation of motion for a linearized material

- Equation of motion:

$$\rho\ddot{\mathbf{u}}(\mathbf{x}) = \int \mathbf{C}(\mathbf{x},\mathbf{q})(\mathbf{u}(\mathbf{q}) - \mathbf{u}(\mathbf{x}))\, dV_{\mathbf{q}} + \mathbf{b}(\mathbf{x})$$

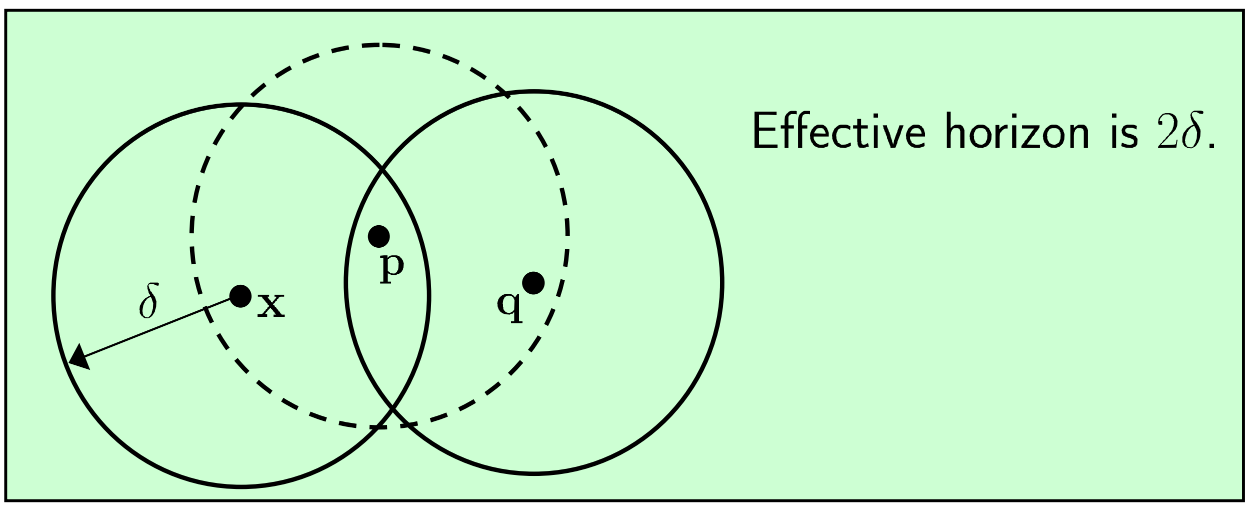
  where

$$\mathbf{C}(\mathbf{x},\mathbf{q}) = \int \big(\underline{\mathbb{K}}[\mathbf{x}]\langle\mathbf{p}-\mathbf{x},\mathbf{q}-\mathbf{x}\rangle - \underline{\mathbb{K}}[\mathbf{p}]\langle\mathbf{x}-\mathbf{p},\mathbf{q}-\mathbf{p}\rangle + \underline{\mathbb{K}}[\mathbf{q}]\langle\mathbf{x}-\mathbf{q},\mathbf{p}-\mathbf{q}\rangle\big)\, dV_{\mathbf{p}}$$

- Similar structure to linear *bond-based* equation of motion but $\mathbf{C}$ has different symmetry:

$$\mathbf{C}(\mathbf{q},\mathbf{x}) = \mathbf{C}^T(\mathbf{x},\mathbf{q}) \qquad \text{...state-based}$$

$$\mathbf{C}(\mathbf{q},\mathbf{x}) = \mathbf{C}(\mathbf{x},\mathbf{q}) \quad \text{and} \quad \mathbf{C}(\mathbf{x},\mathbf{q}) = \mathbf{C}^T(\mathbf{x},\mathbf{q}) \qquad \text{...bond-based}$$



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Stability of a jump perturbation

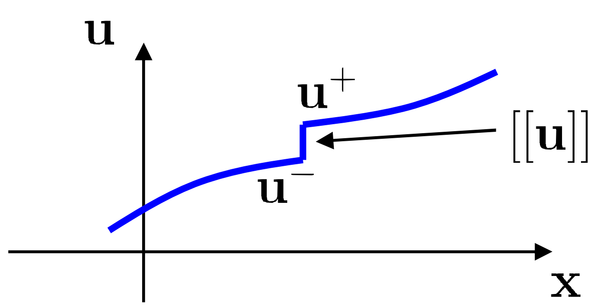
- Write the linearized equation of motion as:

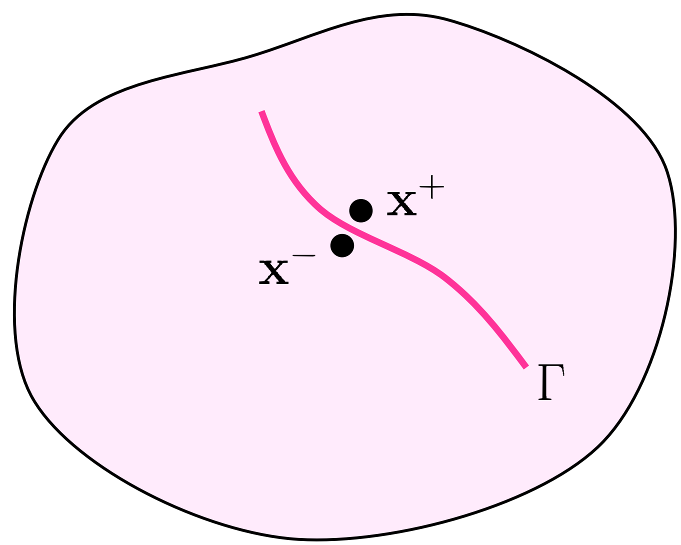
$$\rho\ddot{\mathbf{u}}(\mathbf{x}) = \int \mathbf{C}(\mathbf{x},\mathbf{q})\mathbf{u}(\mathbf{q})\; dV_{\mathbf{q}} - \mathbf{P}(\mathbf{x})\mathbf{u}(\mathbf{x}) + \mathbf{b}(\mathbf{x})$$

  where $\mathbf{P}$ is the symmetric tensor defined by

$$\mathbf{P}(\mathbf{x}) = \int \mathbf{C}(\mathbf{x},\mathbf{q})\; dV_{\mathbf{q}} = \int\int \underline{\mathbb{K}}[\mathbf{x}]\langle \mathbf{p}-\mathbf{x}, \mathbf{q}-\mathbf{x}\rangle\; dV_{\mathbf{p}}\; dV_{\mathbf{q}}$$

- Consider a small superposed displacement field $\mathbf{u}$ containing a jump across a surface Γ.
- Define $[[\mathbf{u}]] = \mathbf{u}^{+} - \mathbf{u}^{-}$.





Stability of a jump perturbation, ctd.

- Write the equation of motion on each side of the jump $(\mathbf{b} = \mathbf{0})$:

$$\rho \ddot{\mathbf{u}}^+ = \int \mathbf{C}(\mathbf{x}^+, \mathbf{q})\mathbf{u}(\mathbf{q})\, dV_{\mathbf{q}} - \mathbf{P}(\mathbf{x}^+)\mathbf{u}^+$$

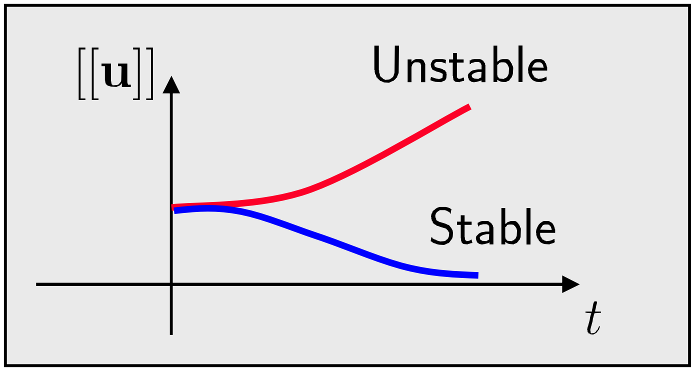
$$\rho \ddot{\mathbf{u}}^- = \int \mathbf{C}(\mathbf{x}^-, \mathbf{q})\mathbf{u}(\mathbf{q})\, dV_{\mathbf{q}} - \mathbf{P}(\mathbf{x}^-)\mathbf{u}^-$$

- $\mathbf{C}$ and $\mathbf{P}$ are continuous. Subtract.

$$\rho[[\ddot{\mathbf{u}}]] = -\mathbf{P}[[\mathbf{u}]]$$

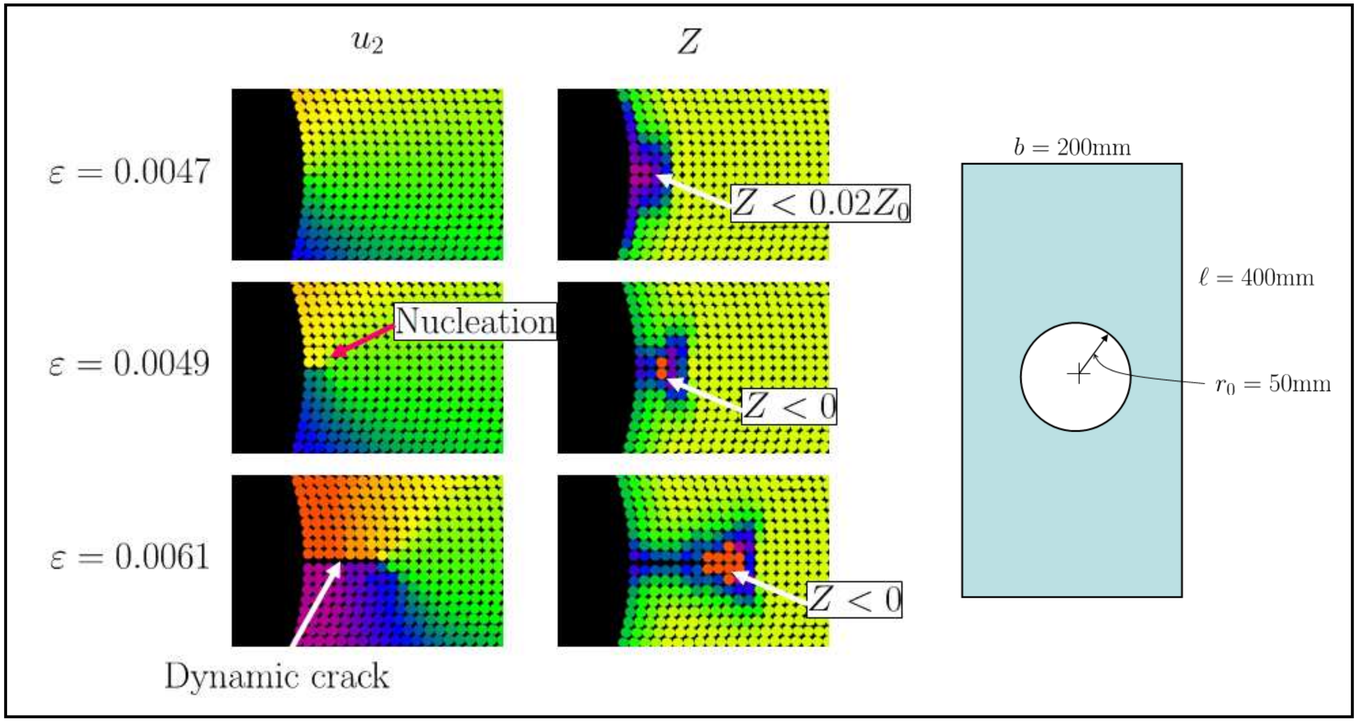
$$\rho[[\ddot{\mathbf{u}}]] \cdot [[\mathbf{u}]] = -\mathbf{P}\big|[[\mathbf{u}]]\big|^2$$

- The jump grows if $[[\ddot{\mathbf{u}}]] \cdot [[\mathbf{u}]] > 0$. This can happen if $\mathbf{P}$ has a negative eigenvalue.



Crack nucleation condition

- Let the eigenvalues of $\mathbf{P}(\mathbf{x})$ be denoted $\{P_1, P_2, P_3\}$ and define the *stability index* by
$$Z(\mathbf{x}) = \min\{P_1, P_2, P_3\}.$$
- If $Z(\mathbf{x}) < 0$ then a crack can nucleate at $\mathbf{x}$.
- $Z(\mathbf{x})$ depends only on the material properties at $\mathbf{x}$.



Materials with a damage variable

- Assume there is a scalar *damage state* $\underline{\phi}$ such that

$$\psi = \psi(\underline{\mathbf{Y}}, \theta, \underline{\phi}) \qquad \text{and} \qquad \dot{\underline{\phi}} \geq 0.$$

Damage is irreversible.

- Repeat C-N argument to find that we still have (for $h = r = 0$)

$$\underline{\mathbf{T}} = \psi_{\underline{\mathbf{Y}}} \qquad \text{and} \qquad \eta = -\psi_\theta$$

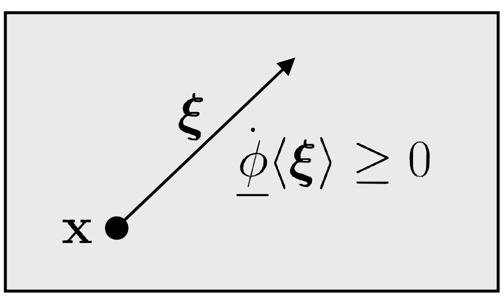
but now also have

$$\psi_{\underline{\phi}} \leq 0$$

and

$$\dot{\eta} = \frac{\dot{\psi}_d}{\theta} \qquad \text{where} \qquad \dot{\psi}_d := -\psi_{\underline{\phi}} \bullet \dot{\underline{\phi}}.$$

Energy dissipation



Damage evolution laws

- $\underline{\phi}\langle\boldsymbol{\xi}\rangle$ is the damage in bond $\boldsymbol{\xi}$ (at some $\mathbf{x}$), determined by a damage evolution law:

$$\underline{\phi} = \underline{D}(\underline{\mathbf{Y}}, \underline{\dot{\mathbf{Y}}}, \dots)$$

- If $\underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle = \mathbf{0}$ whenever $\underline{\phi}\langle\boldsymbol{\xi}\rangle = 1$, the material has *strong* damage dependence (otherwise *weak*).

Damage evolution laws
Example: bond breakage

- Define the bond extension state by

$$\underline{e}\langle\boldsymbol{\xi}\rangle = |\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle| - |\boldsymbol{\xi}|.$$

- Suppose

$$\underline{D}\langle\boldsymbol{\xi}\rangle = H(\underline{e}_0\langle\boldsymbol{\xi}\rangle, e_b)$$

where H=Heaviside step function and

$$\underline{e}_0\langle\boldsymbol{\xi}\rangle = \max_t \langle\boldsymbol{\xi}\rangle.$$

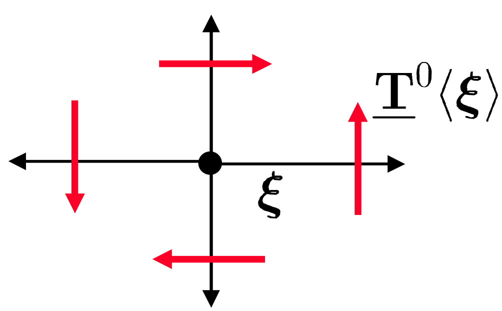
Damage in bond $\boldsymbol{\xi}$ jumps from 0 to 1 when its elongation exceeds the critical elongation e_b.

Damage in a constitutive model: have to be consistent with nonpolarity

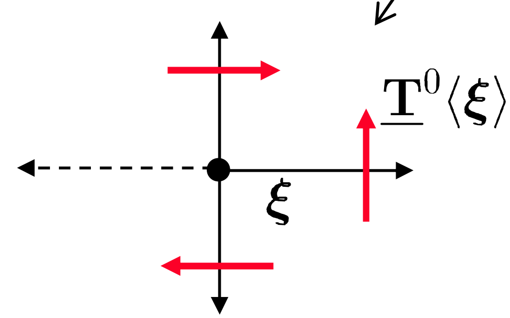
- Cannot in general do the following with a nonordinary material model $\underline{\mathbf{T}}^0$:

$$\underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle = (1 - \underline{\phi}\langle\boldsymbol{\xi}\rangle)\underline{\mathbf{T}}^0\langle\boldsymbol{\xi}\rangle$$

  because the resulting model may fail to be nonpolar.



Four typical bonds in a nonordinary material



Breaking a bond results in a net moment – No longer nonpolar

Damage in a constitutive model: strong damage

- Suppose we have an elastic material with strain energy function $W = W^0(\underline{e})$. Then

$$\underline{\mathbf{T}}^0 = W_{\underline{\mathbf{Y}}} = W^0_{\underline{e}}\underline{\mathbf{M}}, \qquad \underline{\mathbf{M}} = \frac{\underline{\mathbf{Y}}}{|\underline{\mathbf{Y}}|}$$

- Define a material by

$$\psi(\underline{\mathbf{Y}}, \underline{\phi}) = W^0((1-\underline{\phi})\underline{e}).$$

- Then

$$\underline{\mathbf{T}} = (1-\underline{\phi})\underline{\mathbf{T}}^0$$

- Each bond has its force reduced by $1 - \underline{\phi}\langle\boldsymbol{\xi}\rangle$.
- ψ is still objective so model is still nonpolar.

Damage in a constitutive model: separable damage

- Start with $W = W^0(\underline{\mathbf{Y}})$. Then

$$\underline{\mathbf{T}}^0 = W^0_{\underline{\mathbf{Y}}}$$

- Define a material by

$$\psi(\underline{\mathbf{Y}}, \underline{\phi}) = \Phi(\underline{\phi}) W^0(\underline{\mathbf{Y}})$$

  where

$$\Phi(\underline{\phi}) = \frac{1}{V} \int_{\mathcal{H}} (1 - \underline{\phi}\langle \boldsymbol{\xi} \rangle)^2 \, dV_{\boldsymbol{\xi}}.$$

- Find

$$\underline{\mathbf{T}} = \Phi(\underline{\phi}) \underline{\mathbf{T}}^0$$

- All the bond forces are multiplied by the same $\Phi(\underline{\phi})$.

Changing the length scale in a material model

- Suppose we want to change the horizon from δ_0 to δ_1. Require the rescaled energy to be the same as the original W_0 if the deformation is homogeneous.

- The rescaled material model is

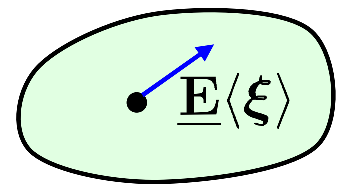
$$W_1(\underline{\mathbf{Y}}) = W_0(\underline{\mathbf{E}})$$

  where $\underline{\mathbf{E}}$ is a state defined by

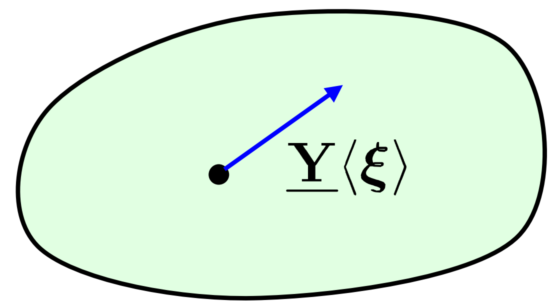
$$\underline{\mathbf{E}}\langle\boldsymbol{\xi}\rangle = \frac{\delta_0}{\delta_1}\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle$$

- Can show the force state scales according to

$$\underline{\mathbf{T}}_1(\underline{\mathbf{Y}}) = \left(\frac{\delta_1}{\delta_0}\right)^4 \underline{\mathbf{T}}_0(\underline{\mathbf{E}})$$



Original, δ_0



Rescaled, δ_1

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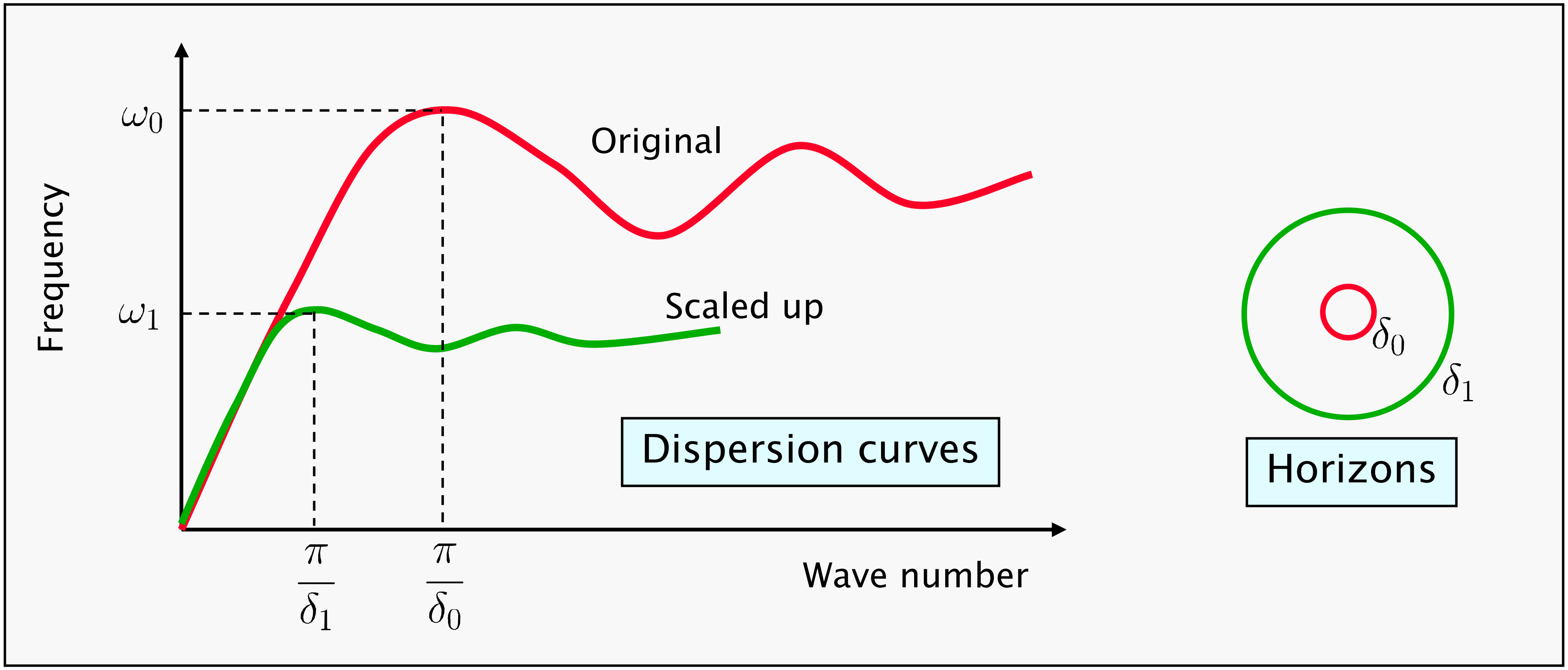
Changing the length scale also changes the time scale

- Removing the small length scale also removes the high frequencies that characterize that length scale.

$$\delta_1 > \delta_0 \implies \omega_1 < \omega_0$$



Peridynamic stress tensor

In any peridynamic body, we can define a tensor field $\boldsymbol{\nu}$ such that:

- The force per unit area at $\mathbf{x}$ through a plane with normal $\mathbf{n}$ is

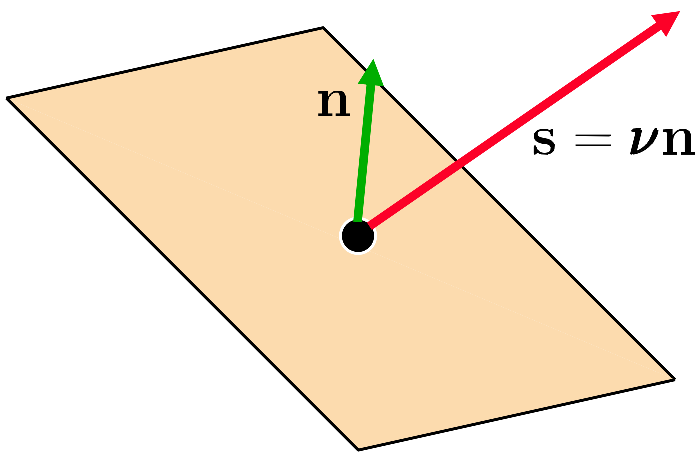
$$\mathbf{s} = \boldsymbol{\nu}(\mathbf{x})\mathbf{n}$$

- The peridynamic equation of motion can be written as

$$\rho\ddot{\mathbf{u}} = \operatorname{div}\boldsymbol{\nu} + \mathbf{b}$$

i.e.,

$$\operatorname{div}\boldsymbol{\nu}(\mathbf{x}) = \int \mathbf{f}(\mathbf{x}', \mathbf{x})\, dV_{\mathbf{x}'}$$



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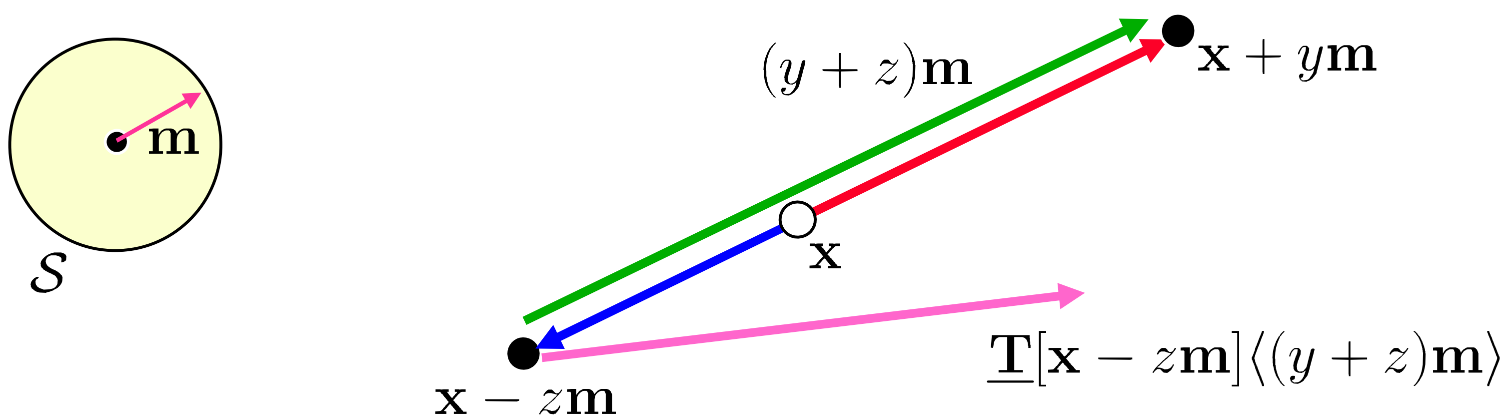
Peridynamic stress tensor, ctd.

- The peridynamic stress tensor is given by

$$\boldsymbol{\nu}(\mathbf{x}) = \int_{\mathcal{S}} \int_0^\infty \int_0^\infty (y+z)^2 \, \big((\underline{\mathbf{T}}[\mathbf{x}-z\mathbf{m}]\langle (y+z)\mathbf{m}\rangle) \otimes \mathbf{m} \big) \, dz \, dy \, d\Omega_{\mathbf{m}}$$

where $\mathcal{S}$ is the unit sphere and Ω is solid angle.

- $\boldsymbol{\nu}$ sums up the forces in bonds that go through $\mathbf{x}$.



Convergence of peridynamics to the standard theory

Suppose the deformation is twice continuously differentiable. If the horizon is small, the deformation state is well approximated by

$$\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle \approx (\nabla\mathbf{y})\boldsymbol{\xi}$$

so we can write

$$W(\underline{\mathbf{Y}}) \approx W_c(\nabla\mathbf{y})$$

and it can be proven that

$$\boldsymbol{\nu} \approx \frac{\partial W_c}{\partial \nabla\mathbf{y}}$$

so $\boldsymbol{\nu}$ is basically a Piola-Kirchhoff stress tensor in a classical hyperelastic solid.